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Vacuum orbit and spontaneous symmetry breaking in hyperbolic sigma-models

A. Duncan, M. Niedermaier, E. Seiler

Department of Physics
100 Allen Hall, University of Pittsburgh
Pittsburgh, PA 15260, USA

Laboratoire de Mathematiques et Physique Theorique
CNRS/UMR 6083, Université de Tours
Parc de Grandmont, 37200 Tours, France

Max-Planck-Institut für Physik
Föhringer Ring 6
80805 München, Germany

Abstract

We present a detailed study of quantized noncompact, nonlinear $\mathrm{SO}(1,N)$ sigma-models in arbitrary space-time dimensions $D \geq 2$, with the focus on issues of spontaneous symmetry breaking of boost and rotation elements of the symmetry group. The models are defined on a lattice both in terms of a transfer matrix and by an appropriately gauge-fixed Euclidean functional integral. The main results in all dimensions ≥ 2 are: (i) On a finite lattice the systems have infinitely many nonnormalizable ground states transforming irreducibly under a nontrivial representation of $\mathrm{SO}(1,N)$; (ii) the $\mathrm{SO}(1,N)$ symmetry is spontaneously broken. For $D = 2$ this shows that the systems evade the Mermin-Wagner theorem. In this case in addition: (iii) Ward identities for the Noether currents are derived to verify numerically the absence of explicit symmetry breaking; (iv) numerical results are presented for the two-point functions of the spin field and the Noether current as well as a new order parameter; (v) in a large N saddle-point analysis the dynamically generated squared mass is found to be negative and of order $1/(V \ln V)$ in the volume, the 0-component of the spin field diverges as $\sqrt{\ln V}$, while $\mathrm{SO}(1,N)$ invariant quantities remain finite.

1. Introduction

Noncompact nonlinear sigma-models occur in a variety of contexts. They are ubiquitous in the dimensional reduction of (super)-gravity theories, which provided the main incentive for the study of their quantum properties [1] – [7]. Motivated by structural similarities they were also used as a test-bed for renormalization and symmetry aspects of quantum gravity [9, 10]. The two-dimensional versions are in addition relevant for the theory of disordered systems and localization, see e.g. [12, 13, 14, 15, 16].

The most intriguing aspect of noncompact sigma-models is the apparent clash between symmetry and unitarity: the Lagrangian is invariant under a finite dimensional – hence nonunitary – representation of the group, while the physical Hilbert space (or at least a sizeable subspace of it) is expected to carry a unitary and hence infinite dimensional representation of the group, apparently not accounted for by the field content of the system. This is particularly puzzling in the vacuum sector, where in the 2-dimensional versions Coleman's theorem [17] seems to preclude spontaneous symmetry breaking even for a noncompact group. Indeed both perturbation theory and large N techniques typically expand around an invariant Fock vacuum in an indefinite metric state space [8, 1, 5]. Its positive metric subspace, however, then carries no remnant of the original noncompact symmetry and looks more like that of a compact model.

A recent detailed study of the 1-dimensional hyperbolic spin chain [18] showed how in that system the clash is avoided: there are infinitely many nonnormalizable ground states transforming under an irreducible representation of the group. On the one hand this entails that the symmetry is spontaneously broken at the level of (certain) correlation functions. On the other hand, by a change of scalar product to the one induced by the Osterwalder-Schrader reconstruction, the above representation rotating the ground states into each other can be made unitary. The price to pay is that the reconstructed Hilbert space is nonseparable and that the unitarity of the representation only extends to a 'large' but proper subspace of it. One of the goals of the present paper is to investigate the extent to which this picture of the 'ground state orbit' generalizes to the field theoretical case.

More generally our focus is on issues of spontaneous symmetry breaking of non-compact (boost) and compact (rotation) symmetries. The starting point is a lattice construction of the models, using both the transfer matrix formalism and the Euclidean functional integral. In either case the infinite volume of the symmetry group requires modifications compared to the setting for a compact symmetry group. The transfer operator is no longer trace class even in finite volume and the functional integral needs to be gauge fixed. Two specific gauge-fixing schemes (a translationally invariant scheme in which the zero-momentum mode of the transverse spin fields is set to zero, and a fixed-spin gauge) are used, the first of which is convenient for numerical simulations while the second one allows one to relate the transfer matrix to the functional integral. Once properly defined (Section 2) the systems are studied by a combination of group theoretical techniques (Section 3), numerical simulations (Section 4), and a large N saddle-point

analysis (Section 5). Our main results in generic dimensions $D \geq 2$ are:

- On a finite spatial lattice the noncompact models are shown to have infinitely many non-normalizable ground states transforming irreducibly under $\mathrm{SO}(1,N)$ – in sharp contrast to the unique ground state of the $\mathrm{SO}(1{+}N)$ models.
- Spontaneous symmetry breaking occurs in all dimensions $D \geq 2$.

As described, the symmetry breaking is surprising in dimension $D = 2$; a case which we therefore investigated in more detail with the following results:

- A new 'Tanh' order parameter is used to probe the spontaneous breaking of the boost symmetries, bypassing problems with the usual 'hysteresis criterion'.
- Quadratic Ward identities for the Noether currents are derived (including finite volume corrections) and used to verify numerically the disappearance of explicit breaking of the boost and rotation symmetries with increasing volume.
- Numerical results are presented for the two-point functions of the spin fields and of the Noether current, as well as for the 'Tanh' order parameter, which show spontaneous symmetry breaking.
- In a large N saddle-point analysis (starting from the gauge fixed functional integral) the dynamically generated squared mass is found to be negative and of order $1/(V \ln V)$ in the volume V, the 0-component of the spin field diverges as $\sqrt{\ln V}$ while $\mathrm{SO}(1,N)$ invariant combinations remain finite.

In addition we point out certain subtleties, related once more to the noncompactness of the symmetry group, without attempting definite answers here. One of them concerns the inapplicability of standard theorems in $D = 2$ (Mermin-Wagner, and refinements thereof) to argue that the maximal compact $\mathrm{SO}(N)$ subgroup singled out by the gauge fixing is not spontaneously broken; see Section 2.2 for a discussion. For any $D \geq 2$ another subtle point is the reconstruction of a Hilbert space, a transfer operator, a normalizable ground state and a representation of the symmetry group commuting with it from the infinite volume correlation functions via an Osterwalder-Schrader reconstruction; see Section 3.5.

The paper is organized as follows: in the next section we introduce the ingredients of a lattice construction of the systems (transfer matrix and functional integral) and pose the questions we wish to address. In Section 3 we derive the a structural characterization of the ground states of the finite lattice systems in $D \geq 2$ and prove that whenever a thermodynamic limit exists it shows spontaneous symmetry breaking. Sections 4 and 5 are devoted to the $D = 2$ model, and contain the Monte-Carlo study of the $\mathrm{SO}(1,2)$ models and the large N saddle-point analysis, respectively. Some technical material on the harmonic analysis of functions on the target space and the finite volume corrections to the Ward identities are relegated to Appendices A and B, respectively.

2. Lattice construction

We consider the hyperbolic SO$(1,N)$ nonlinear sigma-models with $N \geq 2$ defined on a D-dimensional Euclidean lattice, $\Lambda \subset \mathbb{Z}^D$, with $D = d+1 \geq 2$. The systems are defined on finite lattices with the thermodynamic limit $\Lambda \to \mathbb{Z}^D$ to be taken later on. We divide the lattice into time slices $\Lambda_t = \{(x,t) \mid 1 \leq x_\mu \leq L_s\,,\ \mu = 1,\ldots,d\} \simeq \{1,\ldots,L_s\}^d$ of $|\Lambda_t| = L_s^d$ lattice points. The Euclidean time t ranges from 0 to $L_t - 1$, so that $|\Lambda| = L_t L_s^d$ is the total lattice volume. The dynamical variables ('spins') are denoted by n_x, $x \in \Lambda$; those in a given time slice are written alternatively as n_x, $x \in \Lambda_t$ or as $n_{x,t}$, $x \in \{1,\ldots,L\}^d$. The spins take values in $\mathbb{H}_N = \{n \in \mathbb{R}^{1,N} \,|\, n \cdot n = +1,\, n^0 > 0\}$. The bilinear form (dot product) is $a \cdot b = a^0 b^0 - a^1 b^1 - \ldots a^N b^N =: a^0 b^0 - \vec{a}\vec{b}$, with $\vec{a} = (a^1,\ldots,a^N)$. We take as our basic lattice action

$$S_0[n] = \beta \sum_{x,\mu} (n_x \cdot n_{x+\hat{\mu}} - 1)\,, \tag{2.1}$$

where $\hat{\mu}$ is the unit vector in the positive μ-direction (with $\mu = 1,\ldots,D$, and the boundary conditions specified later). Since $n \cdot n' \geq 1$ for all $n, n' \in \mathbb{H}_N$ the action is normalized such that $S_0[n] \geq 0$.

We use the connected component of the identity SO$_0(1,N)$ (which preserves both sheets of the cone $a \cdot a = 0$) thoughout. Slightly simplifying (and abusing) the notation we shall always write SO$(1,N)$:= SO$_0(1,N)$ for it. The hyperboloid $\mathbb{H}_N$ can then also be viewed as a globally symmetric space SO$(1,N)$/SO(N) for any one of the maximal compact SO(N) subgroups. We shall use the stabilizer group SO$^\uparrow(N)$ of the vector $n^\uparrow = (1,0\ldots,0)$ throughout. Concretely this amounts to a parametrization of the spins as $n = (\xi, \sqrt{\xi^2-1}\,\vec{s})$, where $\xi \geq 1$ is a noncompact variable and $\vec{s} \in S^{N-1}$ is a conventional compact spin. Note that this provides a global parametrization of $\mathbb{H}_N$. The invariant products entering the action then read

$$n_x \cdot n_{x+\hat{\mu}} = \xi_x \xi_{x+\hat{\mu}} - (\xi_x^2 - 1)^{1/2} (\xi_{x+\hat{\mu}}^2 - 1)^{1/2} \vec{s}_x \cdot \vec{s}_{x+\hat{\mu}}\,, \tag{2.2}$$

We write $S_0[\xi, s]$ for the action in this parametrization. It can be viewed as that of a spherical S^{N-1} sigma-model coupled in a non-polynomial way to the additional non-compact field ξ_x. The invariant measure $d\Omega(n) := 2d^{N+1}n\,\delta(n\cdot n - 1)\theta(n^0)$ factorizes according to

$$\int d\Omega(n) = \int_1^\infty d\xi (\xi^2 - 1)^{N/2-1} \int_{S^{N-1}} dS(\vec{s})\,. \tag{2.3}$$

2.1 Definition of the transfer matrix and functional integral

The dynamics of the lattice system is defined in terms of the transfer operator $\mathbb{T}$ which transports lattice configurations from one time slice to the next. The square integrable wave functions $\psi(n) = \psi(n_x,\, x \in \Lambda_t)$ depending on the spins in some time slice Λ_t form the Hilbert space $L^2(\mathbb{H}_N^{L_s^d})$ with respect to the product of the canonical invariant measure. Since we usually keep L_s fixed we simply write L^2 for this Hilbert space once the number of spatial dimensions d is clear from the context. The transfer operator $\mathbb{T}$ acts on L^2 as an integral operator via

$$\begin{aligned}(\mathbb{T}\psi)(n) &= \int \prod_{x\in\Lambda_t} d\Omega(n'_x)\, \mathcal{T}_\beta(n,n';1)\psi(n')\,, \\ \mathcal{T}_\beta(n,n';1) &= D_{\beta,N}^{-L_s} \exp\left\{-\beta\sum_{x\in\Lambda_t}\left[n_x\cdot n'_x + \frac{1}{2}n_x\cdot n_{x+\hat{1}} + \frac{1}{2}n'_x\cdot n'_{x+\hat{1}} - 2\right]\right\}.\end{aligned} \tag{2.4}$$

The normalization constant $D_{\beta,N}$ is introduced for later use; it sets the overall scale in that $0 < \mathcal{T}_\beta(n,n';1) \leq D_{\beta,N}^{-L_s}$ for all configurations. The variables in $(\mathbb{T}\psi)(n)$ can then naturally be associated with the time slice Λ_{t+1}. Indeed upon iteration of (2.4) one obtains

$$(\mathbb{T}^t\psi)(n) = \int \prod_{x\in\Lambda_0} d\Omega(n'_x)\, \mathcal{T}_\beta(n,n';t)\psi(n')\,, \tag{2.5}$$

where we conventionally regard $\mathbb{T}^t$ as a map from time slice Λ_0 to time slice Λ_t. In this interpretation the iterated kernel reads

$$\begin{aligned}\mathcal{T}_\beta\big(n\in\Lambda_t, n\in\Lambda_0;t\big) &= D_{\beta,N}^{-L_s t}\, \exp\left\{\frac{\beta}{2}\sum_{\mu\neq D}\Big(\sum_{x\in\Lambda_0} - \sum_{x\in\Lambda_t}\Big) n_x\cdot n_{x+\hat{\mu}}\right\} \\ &\times \int \prod_{x\in\Lambda_1,\ldots,\Lambda_{t-1}} d\Omega(n_x)\, \exp\left\{-\beta\sum_{\mu,x\in\Lambda_0,\ldots,\Lambda_{t-1}}(n_x\cdot n_{x+\hat{\mu}} - 1)\right\}.\end{aligned} \tag{2.6}$$

For $t = L_D$ and periodic bc in the x_D-direction the last expression clearly resembles the partition function for the action (2.1). The last integration over the variables $n_{x,0} = n_{x,L_2}$, however, would be divergent as the infinite volume of $\mathbb{H}_N$ gets overcounted.

To see this more clearly note that the wave functions $\psi(n)$ carry the following "diagonal action" of $\mathrm{SO}(1,N)$,

$$\rho(A)\psi(n_x, x\in\Lambda_t) = \psi(A^{-1}n_x, x\in\Lambda_t)\,, \quad A\in \mathrm{SO}(1,N)\,. \tag{2.7}$$

In contrast to a lattice system with a compact symmetry group ρ invariant wave functions, i.e. those satisfying $\rho(A)\psi(n) = \psi(n)$, for all $A\in\mathrm{SO}(1,N)$, do *not* lie in the Hilbert space. This is because in the inner product on L^2 one of the integrations factorizes, the

infinite volume of $\mathbb{H}_N$ gets overcounted and the L^2 norm of the wave function diverges. On an integral operator K with kernel $\kappa(n, n')$ the group acts as $K \to \rho(A)^{-1} K \rho(A)$ and thus as $\kappa(n, n') \to \kappa(An, An')$ on the kernels. In particular operators K whose kernels only depend on the invariants $n_x \cdot n_y$ are invariant. Importantly this holds for the iterated transfer operator, i.e.

$$\mathbb{T}^t \circ \rho = \rho \circ \mathbb{T}^t \,, \qquad \forall\, t \in \mathbb{N} \,. \tag{2.8}$$

Since the semigroup $\mathbb{T}^t$, $t \in \mathbb{N}$, describes the evolution of the system in Euclidean time Eq. (2.8) means that the dynamics is $\mathrm{SO}(1, N)$ invariant, as required. On the other hand it also implies that, although $\mathbb{T}$ is bounded, in contrast to the transfer operator of most other lattice systems in finite volume (as we shall see later) it is not trace class (see [18] for the 1-dim. case). As a consequence correlation functions cannot be defined in terms of the usual expressions involving traces. The remedy is to ('gauge') fix the residual ρ symmetry by a variant of the familiar Faddeev-Popov procedure. To the best of our knowledge this gauge-fixing does not seem to have been taken into account in earlier studies, rendering the results somewhat formal. We now first describe this procedure and then outline the relation to the transfer operator.

We used the following two gauge-fixing choices, both of which leave the stability group $\mathrm{SO}^{\uparrow}(N)$ of the vector $n^{\uparrow} = (1, 0, \ldots, 0)$ intact:

1. The noncompact global gauge freedom is eliminated with a translationally invariant gauge choice which sets the zero momentum mode of the transverse sigma fields $\vec{n} := \sqrt{\xi^2 - 1}\vec{s}$ to zero:
$$\sum_{x \in \Lambda} \vec{n}_x = 0 \,. \tag{2.9}$$
In this gauge there is a nontrivial Faddeev-Popov determinant which comes out to be $(\sum_x n_x^0)^N$, see [19] in the compact case. The expectations of a general multilocal observable $\mathcal{O}(\{n\})$ then assume the form
$$\begin{aligned} \langle \mathcal{O} \rangle_{\Lambda,\beta,1} \;=\; & \frac{1}{Z_1(\Lambda, \beta)} \int \prod_x d\Omega(n_x)\, \mathcal{O}(\{n\})\, \delta\Big(\sum_x \vec{n}_x \Big) \\ & \times \exp\Big\{ - S_0[n] + N \ln \sum_x n_x^0 \Big\} \,, \end{aligned} \tag{2.10}$$
where $Z_1(\Lambda, \beta)$ is the partition function normalizing the averages, $\langle \mathbb{1} \rangle_{\Lambda,\beta,1} = 1$.

2. Alternatively, the noncompact global gauge-freedom may be eliminated by freezing a single spin at an arbitrary site x_0 to a conventional fixed value, typically $n_{x_0} = n^{\uparrow}$, where $n^{\uparrow} = (1, 0, \ldots, 0)$ and $x_0 \in \Lambda_0$. In this case there is no Faddeev-Popov factor and the expectation value of a general observable $\mathcal{O}$ is simply
$$\langle \mathcal{O} \rangle_{\Lambda,\beta,2} = \frac{1}{Z_2(\Lambda, \beta)} \int \prod_{x \neq x_0} d\Omega(n_x)\, \mathcal{O}(\{n\})\, e^{-S_0[n]} \,, \tag{2.11}$$

if we assume that the support of the observable does not include x_0 (otherwise an explicit $\delta(n_{x_0}, n^\uparrow)$ factor has to be included). Although the choice of a specific site x_0 would seem to destroy the translational invariance of the theory, the fact that this choice corresponds to a global gauge transformation implies that $\mathrm{SO}(1,N)$ invariant observables are unaffected, and translational invariance still holds for such observables, provided of course this invariance is not explicitly broken by boundary conditions. We consider again periodic boundary conditions (bc) and denote the resulting expectations by $\langle\mathcal{O}\rangle_{\Lambda,\beta,2}$. Invariant observables then should have the same expectations as with the translationally invariant gauge fixing, i.e. $\langle\mathcal{O}\rangle_{\Lambda,\beta,1} = \langle\mathcal{O}\rangle_{\Lambda,\beta,2}$, for $\mathcal{O}$ $\mathrm{SO}(1,N)$ invariant.

We state without proof that the finite volume partition functions Z_1 and Z_2 are well-defined, i.e. the gauge fixing is sufficient to render the integrals finite. Another interesting choice of bc, in the case of the fixed spin gauge, would be periodic bc in the spatial and free bc in the temporal direction. In analogy to the 1-dimensional model the thermodynamic limit of these expectations should be expected to be different from each other, thereby revealing a peculiar kind of 'long-range order'. However to study the issue through numerical simulations would presumably require much larger lattices and a cluster algorithm.

One can view $\langle\ \ \rangle_{\Lambda,\beta,i}$ as linear functionals over the algebra of bounded observables $\mathcal{C}_b$, that is, continuous bounded functions $\mathcal{O}$ of finitely many spins with pointwise addition and multiplication and equipped with the supremum norm, $\mathcal{O} \mapsto \|\mathcal{O}\|$. As such they qualify as states in the statistical mechanics sense: $|\langle\mathcal{O}\rangle| \leq \|\mathcal{O}\|$ and for nonnegative $\mathcal{O}$ the expectation value is nonnegative.

The fixed spin gauge with periodic bc also allows one to make contact with the transfer matrix (2.6). For example

$$\int \prod_{x\in\Lambda_0} d\Omega(n_x)\, \delta(n_{x_0}, n^\uparrow)\, \mathcal{T}_\beta(n, n; L_t) = Z_2(\Lambda, \beta)\,, \tag{2.12}$$

gives the partition function. Here $x_0 \in \Lambda_0$ and $\delta(n, n^\uparrow)$ is the invariant delta-distribution concentrated at $n = n^\uparrow$ with respect to the measure $d\Omega(n)$. Similarly for the expectation of a generic (noninvariant) observable $\mathcal{O}(n_x, n_y)$ located at $x = (x_1, \ldots, x_D)$, $y = (y_1, \ldots, y_D)$, one has

$$\begin{aligned}\langle\mathcal{O}(n_x, n_y)\rangle_{\Lambda,\beta,2} &= \frac{1}{Z_2(\Lambda,\beta)} \int \prod_{z\in\Lambda_0} d\Omega(n_z)\delta(n_{z=x_0}, n^\uparrow) \prod_{x\in\Lambda_{x_D}} d\Omega(n_x) \prod_{y\in\Lambda_{y_D}} d\Omega(n_y) \\ &\times \mathcal{T}_\beta(n_z, n_x; x_D)\mathcal{O}(n_x, n_y)\, \mathcal{T}_\beta(n_x, n_y; y_D - x_D)\mathcal{T}_\beta(n_y, n_z; L_t - y_D)\,. \end{aligned} \tag{2.13}$$

Because of the gauge fixing for finite L_t this is in general not invariant under time translations (nor, for that matter, under space translations). An important exception are $\mathrm{SO}(1,N)$ invariant observables $\overline{\mathcal{O}}$, satisfying $\overline{\mathcal{O}}(An_x, An_y) = \overline{\mathcal{O}}(n_x, n_y)$, for all $A \in$

SO$(1,N)$. In this case Eq. (2.13) simplifies to

$$\langle \overline{\mathcal{O}}(n_x,n_y)\rangle_{\Lambda,\beta,2} = \frac{1}{Z_2(\Lambda,\beta)} \int \prod_{z\in\Lambda_0} d\Omega(n_z)\delta(n_{z=x_0},n^{\uparrow}) \prod_{x\in\Lambda_{x_D}} d\Omega(n_x)$$

$$\times \overline{\mathcal{O}}(n_z,n_x)\,\mathcal{T}_\beta(n_z,n_x;y_D-x_D)\mathcal{T}_\beta(n_x,n_z;L_t+x_D-y_D)\,, \tag{2.14}$$

using the invariance of the integration measure and the convolution property for the kernels (2.6). Here translation invariance in the time direction (and trivially in the spatial direction) is manifest. Eqs. (2.13) and (2.14) generalize straightforwardly to observables $\mathcal{O}$ depending on more than two spins.

Note also that any single SO$(1,N)$ transformation on an observable $\mathcal{O}$ can always be compensated by a change in the gauge fixing condition

$$\langle \rho(A)\mathcal{O}\rangle_{\Lambda,\beta,i} = \langle \mathcal{O}\rangle_{\Lambda,\beta,i}\Big|_{n^{\uparrow}\to A^{-1}n^{\uparrow}}\,, \qquad i=1,2\,. \tag{2.15}$$

If bc other than periodic ones were adopted the bc would likewise be “counter rotated” in (2.15). The issue of spontaneous symmetry breaking we wish to address in the following, of course, asks for the invariance or noninvariance of the expectations under all of SO$(1,N)$ or a continuous subgroup thereof, with the gauge fixing and bc held fixed and $\Lambda\to\mathbb{Z}^D$.

2.2 Ward identities – absence of explicit symmetry breaking

In this context it is important to make sure that no explicit breaking of the symmetry is induced by the gauge fixing. In the thermodynamic limit one expects that the effect of a single fixed mode fades out, but experience with the 1D model [18] shows that such expectations can be misleading. In addition, as simulations are done on a finite lattice a quantitative assessment would be useful. This can be done by deriving Ward identities expressing the SO$(1,N)$ invariance of all but the gauge fixing terms in the functional measure, with the latter giving rise to finite volume corrections of the ‘naive’ Ward identities. In this section we describe the principle of the derivation as well as the ‘naive’ form of the Ward identities which is dimension independent. In contrast the form of the finite volume corrections is dimension dependent and their determination is rather technical. For the case $D=2$ (where the symmetry breaking is surprising) this is done in appendix B. Throughout this section the translation invariant gauge fixing 1 with periodic bc will be used, i.e. Eq. (2.10).

In lattice models with a compact symmetry group the invariance of the functional measure (including the Boltzmann factor) gives rise to Ward identities in a well-known way: implement a local symmetry transformation and expand the functional integral in powers of the gauge parameter(s). Since the total response must vanish the coefficient of

each power must vanish, which gives rise to identities relating correlators of the Noether current to other correlators. In the case at hand the gauge fixing and the associated Faddeev-Popov determinant lead to a non-invariant overall measure. Nevertheless the impact of a local symmetry transformation can be computed and leads to modified Ward identities.

We begin by fixing our conventions for the Noether current. It takes values in the Lie algebra $so(1,N)$, and we normalize the components with respect to the previously used basis t^{ab}, $0 \le a < b \le N$, according to

$$J^{ab}_{\hat\mu}(x) = \beta t^{ab} n_x \cdot n_{x+\hat\mu} = -\beta n_x \cdot t^{ab} n_{x+\hat\mu} = \beta[n^a_x n^b_{x+\hat\mu} - n^b_x n^a_{x+\hat\mu}]\,. \tag{2.16}$$

Their two-point correlators can be decomposed into a transversal, a longitudinal, and a harmonic piece. This is conveniently done in Fourier space

$$\langle J^{ab}_{\hat\mu}(x)\, J^{ab}_{\hat\nu}(y)\rangle_{\Lambda,\beta,i} =: \frac{1}{|\Lambda|}\sum_p e^{-ip\cdot(x-y)}\mathcal{J}^{ab}_{\mu\nu}(p)\,, \tag{2.17}$$

where p runs over the dual lattice, $p_\mu = \frac{2\pi}{L} n_\mu$, $n_\mu = 0,1,\ldots,L_\mu - 1$, $\mu = 1,2$. In order not to clutter the notation we suppress the specifications (Λ,β,i) on the right hand side (remember that $i = 1,2$ refers to the gauge-fixing adopted). The irreducible components $\mathcal{J}^{ab}_T(p)$ (transversal), $\mathcal{J}^{ab}_L(p)$ (longitudinal), and $\mathcal{J}^{ab}_H(p)$ (harmonic) are picked out by acting with the corresponding projectors on $\mathcal{J}^{ab}_{\mu\nu}(p)$, see e.g. [20] for details.

As mentioned earlier, Ward identities now arise from studying the response of a given expectation value under a local symmetry variation $n_x \to \exp(\alpha_x t^{ab}) n_x$, $x \in \Lambda$, performed on all spins. To get the response of the action we prepare ($\epsilon_a := \eta^{aa}$ equals 1 for $a = 0$ and -1 for $a \neq 0$):

$$\begin{aligned}\Big(e^{\alpha_x t^{ab}} n_x\Big)\cdot\Big(e^{\alpha_{x+\hat\mu} t^{ab}} n_{x+\hat\mu}\Big) &= n_x\cdot n_{x+\hat\mu} + \frac{1}{\beta}\Delta_{\hat\mu}\alpha_x\, J^{ab}_{\hat\mu}(x)\\ &- \frac{1}{2}(\Delta_{\hat\mu}\alpha_x)^2\Big(\epsilon_b n^a_x n^a_{x+\hat\mu} + \epsilon_a n^b_x n^b_{x+\hat\mu}\Big) + O(\alpha^3)\,,\end{aligned} \tag{2.18}$$

where $\Delta_{\hat\mu}\alpha_x = \alpha_{x+\hat\mu} - \alpha_x$ and $[(t^{ab})^2 n_x]^c = -\epsilon_b n^a_x \delta^{ac} - \epsilon_a n^b_x \delta^{bc}$ was used. For the change in the Boltzmann factor this gives

$$\begin{aligned}\exp\{-\beta\sum_{x,\mu} n_x\cdot n_{x+\hat\mu}\} \;\longrightarrow\; \exp\{-\beta\sum_{x,\mu} n_x\cdot n_{x+\hat\mu}\}\Big\{1 + \sum_{x,\mu}\alpha_x\,\Delta^*_{\hat\mu} J^{ab}_{\hat\mu}(x)\\ +\frac{1}{2}\sum_{x,\mu;y,\nu}\alpha_x\alpha_y\,\Delta^*_{\hat\mu}J^{ab}_{\hat\mu}(x)\Delta^*_{\hat\nu}J^{ab}_{\hat\nu}(y) + \frac{\beta}{2}\sum_{x,\mu}(\Delta_{\hat\mu}\alpha_x)^2\Big(\epsilon_b n^a_x n^a_{x+\hat\mu} + \epsilon_a n^b_x n^b_{x+\hat\mu}\Big) + O(\alpha^3)\Big\}\,,\end{aligned} \tag{2.19}$$

where $\Delta^*_{\hat\mu} J^{ab}_{\hat\mu}(x) = J^{ab}_{\hat\mu}(x) - J^{ab}_{\hat\mu}(x+\hat\mu)$. Using (2.19) the expansion of the Boltzmann factor to $O(\alpha^2)$ is trivial. Since the product of the invariant measures $\prod_x dn_x \delta(n_x^2 - 1)$

is invariant even under local SO$(1,N)$ rotations, the only non-invariant terms in the functional measure come from the gauge fixing. Focusing on the invariant terms the total response under a local symmetry transformation must vanish. In principle the vanishing of the coefficients of each power in α gives rise to a new identity.

For example, the $O(\alpha)$ terms in the response of $\langle n_y^c\rangle_{\Lambda,\beta,i}$ give rise to the following first order Ward identity

$$\langle \Delta^*_{\hat\mu} J^{ab}_{\hat\mu} n_y^c\rangle_{\Lambda,\beta,i} + \delta_{x,y}\langle (t^{ab}n_y)^c\rangle_{\Lambda,\beta,i} + \text{terms from gauge fixing} = 0\,. \tag{2.20}$$

Replacing n_y^c by with a generic (noninvariant) observable $\mathcal{O}(n_{x_1},\ldots,n_{x_\ell})$ a similar identity arises where the correlator with $\Delta^*_{\hat\mu} J^{ab}_{\hat\mu}$ produces a sum of contact terms. We shall not pursue these first order Ward identities further: in Section 5 we shall verify in a large N analysis that $\langle n_x^a\rangle_{\Lambda,\beta,i}$ diverges as $|\Lambda|\to\infty$. One expects this to hold also at fixed N, in which case already the example (2.20) shows that these first order Ward identities do not necessarily have an interesting thermodynamic limit. This very fact however is worth mentioning, because it shows how the conflict with Coleman's theorem [17] is avoided: the currents simply do not exist in the thermodynamic limit.

More interesting is the second order Ward identity from the response of the partition function itself [20]. The vanishing of the $O(\alpha^2)$ terms requires

$$\frac{1}{2}\sum_{x,\mu,y,\nu} \alpha_x\alpha_y \langle \Delta^*_{\hat\mu} J^{ab}_{\hat\mu}\, \Delta^*_{\hat\nu} J^{ab}_{\hat\nu}\rangle_{\Lambda,\beta,i} + \frac{\beta}{2}\sum_{x,\mu}(\Delta_{\hat\mu}\alpha)^2(\epsilon_b E^a + \epsilon_a E^b)$$
$$+ \text{terms from gauge fixing} = 0\,, \tag{2.21}$$

where $E^a := E^a_{\Lambda,\beta,i} := \langle n_x^a n^a_{x+\hat\mu}\rangle_{\Lambda,\beta,i}$ is the action link variable. The terms induced by the gauge fixing can in principle be computed exactly. They are expected to die out as $\Lambda\to\mathbb{Z}^D$, but the precise form of the correction terms is cumbersome to compute.

As the case $D=2$ is of particular interest the derivation of the finite volume corrections is detailed in appendix B. The extra terms induced by the gauge fixing then turn out to be of order $O(\ln|\Lambda|/|\Lambda|)$ in the limit of large volumes $|\Lambda|$. Converting (2.21) into Fourier space the longitudinal part $\mathcal{J}_L^{ab}(p)$ of the current two-point function appears. The resulting Ward identity generalizes that in the compact models [20] and reads

$$\mathcal{J}_L^{ab}(p) = -\beta(\epsilon_b E^a + \epsilon_a E^b) + O\Big(\frac{\ln|\Lambda|}{|\Lambda|}\Big)\,,\qquad \forall p\neq 0\,,\quad a<b\,, \tag{2.22}$$

On account of the invariance of the vacua under the maximal compact subgroup singled out by the gauge fixing (see Section 2.3) one expects that $E^0\geq E^1 = \ldots = E^N\geq 0$, so that only two distinct cases arise:

$$\begin{aligned}
\mathcal{J}_L^{12}(p) &= +2\beta E^1 + O\Big(\frac{\ln|\Lambda|}{|\Lambda|}\Big)\,, && \text{rotations}\,,\\
\mathcal{J}_L^{01}(p) &= \beta(E^0 - E^1) + O\Big(\frac{\ln|\Lambda|}{|\Lambda|}\Big)\,, && \text{boosts}\,.
\end{aligned} \tag{2.23}$$

All quantities in (2.22), (2.23) of course depend on the specifications (Λ, β, i). The inequality $E^0 \geq E^a$, $a \neq 0$, follows from $n^0 \geq |\vec{n}|$. Combined with the trivial identity $n_x \cdot n_{x+\hat{\mu}} \geq 1$ one gets the stronger bound $E^0 - NE^1 \geq 1$.

The individual E^a cannot be expected to have a finite thermodynamic limit. In Section 5.1 we verify that in the large N expansion both E^0 and E^1 diverge logarithmically with the volume, according to $E^0 \sim \frac{\lambda}{4\pi} \ln |\Lambda|$ and $E^1 \sim \frac{1}{N}\frac{\lambda}{4\pi} \ln |\Lambda|$, where $\lambda = N/\beta$. In contrast the invariant combination $E^0 - NE^1$ approaches the finite constant $1 + \lambda/4$. For the Ward identities (2.22) therefore only the invariant combination is assured to have a finite thermodynamic limit. Nevertheless the Ward identities for the individual components are useful to test quantitatively the degree to which the boost/rotation symmetry is restored on a finite lattice (as far as the dynamics is concerned). Since the current correlator (2.17) and the action link variables E^a are independently measurable quantities in a Monte-Carlo simulation, validity of the identities (2.23) also provides a good test on the simulations for given lattice size and boundary conditions. We report the results of such a test in Section 4.1.

2.3 Spontaneous symmetry breaking and 'Tanh' order parameter

Even the very notion of spontaneous symmetry breaking in the noncompact models requires a little thought. The conventional analysis of spontaneous symmetry breaking asks if there is a local observable having a noninvariant expectation value if we either

(a) fix symmetry breaking boundary conditions and then take the thermodynamic limit, or
(b) add a symmetry breaking term like a magnetic field h to the action, take the thermodynamic limit and then turn the symmetry breaking term off.

In the second picture spontaneous symmetry breaking amounts to a hysteresis effect. In a model with a compact symmetry group then the one-sided derivatives $h \to 0^+$ and $h \to 0^-$ exist, but are different. This way of looking at spontaneous symmetry breaking, however, does not readily generalize to the boost symmetries in the noncompact sigma-models because the field h has to serve double duty – as a regulator and as a probe for symmetry breaking. For invariant observables it is clear that a nonzero field is needed in order to (potentially) produce a normalizable measure even in finite volume. For noninvariant observables coupling to a magnetic field may or may not not render the finite volume expections finite. Indeed, a typical coupling would add a term of the form $h \sum_x n_x^0$ to the action (2.1). However for $h < 0$ then already the finite volume averages fail to exist. The $h \to 0^+$ derivative is expected to be convergent for $D \geq 3$ and divergent for $D = 2$.

Indeed, since the first version of this paper was posted, an interesting result by Spencer and Zirnbauer appeared [21], in which it was shown that in $D \geq 3$ the expectations $\langle n^0 \rangle_{\Lambda,\beta,h}$ defined without gauge fixing and with a positive magnetic field h can be

bounded by a constant (independent of h and $|\Lambda|$) for all $\beta \geq 3/2$ and $|\Lambda|h \geq 1$. Thus a 'one-sided hysteresis criterion' here signals spontaneous symmetry breaking in the thermodynamic limit.

The case $D = 2$ will be discussed in more detail in section 2.4; in this case even the 'one-sided hysteresis criterion' is expected to fail. In $D = 2$ many authors found that $\langle n^0 \rangle$ diverges in the thermodynamic limit, based on an (un-gauge fixed) large N expansion. This amounts to some vestige of the large fluctuations that are responsable for the symmetry restoration in compact and abelian models. However since $\langle An^0 \rangle, A \in \mathrm{SO}(1,N)$, diverges likewise one can only conclude that the symmetry breaking cannot be seen on this particular observable.

The approach adopted here is somewhat different. The gauge fixed functional integrals (2.10) and (2.11) provide a complete definition of the systems in finite volume, both for invariant and for non-invariant observables. The regulator (gauge fixing) is decoupled from whatever probe is used for the symmetry breaking. Spontaneous symmetry breaking can then be discussed without appeal to a 'one-sided hysteresis criterion' and for all $D \geq 1$. The criterion we propose is simply that there exist noninvariant observables $\mathcal{O}$ for which the thermodynamic limit exists and for which

$$\lim_{\Lambda \to \mathbb{Z}^D} \langle \mathcal{O}(An) \rangle_{\Lambda,\beta,i} \neq \lim_{\Lambda \to \mathbb{Z}^D} \langle \mathcal{O}(n) \rangle_{\Lambda,\beta,i} \,, \quad \text{for some } \; A \in \mathrm{SO}(1,N)\,. \tag{2.24}$$

We should remark that (2.24) for a boost A signals also breaking of the compact subgroup obtained by conjugating $\mathrm{SO}^{\uparrow}(N)$ with A. Spontaneous symmetry breaking then basically follows from the nonamenability of the group $\mathrm{SO}(1,N)$. For convenience we recall the definition here:

A Lie group G is called *amenable* if there exists a left invariant positive linear functional ("a mean") on $\mathcal{C}_b(G)$, the space (and commutative C^*-algebra with unit) of bounded continuous functions on G equipped with the sup-norm. All abelian and all compact Lie groups are amenable. Conversely, G is called *nonamenable* if no such mean exists. All noncompact semisimple nonabelian Lie groups are known to be nonamenable.

In the present context the nonamenability of $\mathrm{SO}(1,N)$ implies that there has to be bounded continuous functions of one spin, say at the origin, whose infinite volume expectation values are not invariant under the group. The precise form of this result is described in Theorem 3 of section 3.

In [18] we identified for the 1D model a useful example of such a function, the so-called 'Tanh' order parameter. As explained in Section 2.2 the 'one-sided hysteresis criterion' to describe spontaneous symmetry breaking cannot readily be used for $D \leq 2$. The 'Tanh' order parameter, on the other hand, does not require the introduction of an external field; the gauge fixing or the boundary conditions single out the direction of symmetry breaking and the maximally compact subgroup $\mathrm{SO}^{\uparrow}(N)$ that remains unbroken is the stability group of $n^{\uparrow}$. This construction readily generalizes to all $D \geq 1$:

For a spacelike unit vector e we define

$$
\begin{aligned}
T_e(n) &:= \tanh(e\cdot n)\,, \qquad e\cdot e=-1\,,\\
\overline{T}_q(\xi) &:= \int_{\mathrm{SO}^{\uparrow}(N)} d\mu(A)\, T_e(An) \\
&= \int_{\mathrm{SO}^{\uparrow}(N)} d\mu(A)\, \tanh\Big(\xi\sqrt{q^2-1} - q\sqrt{\xi^2-1}\,\vec{e}_0\cdot A\vec{s}\Big)\,,
\end{aligned}
\tag{2.25}
$$

where $d\mu(A)$ is the normalized Haar measure on $\mathrm{SO}^{\uparrow}(N)$. After the group averaging the observable only depends on $\xi := n^{\uparrow}\cdot n$ and $q := \sqrt{n^{\uparrow}\cdot e+1}$. Here we parameterized n and e as $n = (\xi, \sqrt{\xi^2-1}\vec{s})$, $\vec{s}^2 = 1$ and $e = (\sqrt{q^2-1}, q\vec{e}_0)$, $\vec{e}_0^{\,2} = 1$. This observable of course remains finite for $\Lambda \to \mathbb{Z}^D$ even if $\langle n^0\rangle_{\Lambda,\beta,i}$ diverges. More importantly it is designed to be a good indicator for 'spontaneous' symmetry breaking already in finite volume. The criterion (2.24) for spontaneous symmetry breaking becomes for all $D \geq 1$: $\langle T_e(n)\rangle_{\infty,\beta,i} \neq \langle T_e(An)\rangle_{\infty,\beta,i}$ for some $A \in \mathrm{SO}(1,N)$. Since by Section 2.3 the finite volume average in itself effects the $\mathrm{SO}^{\uparrow}(N)$ average this is equivalent to $T(q) := \langle \overline{T}_q(n^0)\rangle_{\infty,\beta,i}$ having a nontrivial dependence on q. Clearly $|T(q)| \leq 1$ and $T(1) = 0$, by the $\mathrm{SO}^{\uparrow}(N)$ invariance.

Typically a nonzero value for $\langle T_e(n)\rangle_{\Lambda,\beta,i}$ at some $q > 1$ is numerically easy to detect. In order to view this as a signal for spontaneous symmetry breaking one has to exclude that this value decays to zero as $\Lambda \to \mathbb{Z}^D$. Since by a convexity argument one expects

$$
\langle T_e(n^0)\rangle_{\Lambda,\beta,i} \geq \overline{T}_q(\langle n^0\rangle_{\Lambda,\beta,i}) \geq \overline{T}_q\Big(\sup_{\Lambda}\langle n^0\rangle_{\Lambda,\beta,i}\Big)\,,
\tag{2.26}
$$

every nontrivial $\langle n^0\rangle_{\Lambda,\beta,i}$ will thus provide a lower bound on the measured $\langle T_e(n)\rangle_{\Lambda,\beta,i} > 0$, which therefore cannot decay to zero as $\Lambda \to \mathbb{Z}^D$. For $D = 2$ we shall find later in the large N limit that $T(q)$ is in fact a strictly increasing function of q approaching 1 for $q \to \infty$. Specifically one has $\overline{T}_q(\xi) \sim \tanh(\bar{n}\sqrt{q^2-1})$, with $\bar{n}$ given by Eq. (5.6) below. For $N = 2$ the same monotone increasing behavior is found in numerical simulations, see Section 4.3.

2.4 Unbroken $\mathrm{SO}^{\uparrow}(N)$ invariance in $D = 2$?

In two dimensions an additional subtlety arises from the Mermin-Wagner theorem [22] and its refinements [23, 24]. Whether in the fixed spin gauge or in the translation invariant gauge, the system has a residual $\mathrm{SO}^{\uparrow}(N)$ invariance and can be viewed as a $\mathrm{O}(N)$ vector model with fluctuating length of the spin vectors. In $D = 2$, at first sight it may seem obvious that this compact symmetry cannot be spontaneously broken, due to the mentioned theorems.

On closer inspection, however, the situation is not quite as simple. The above mentioned theorems on the absence of spontaneous symmetry breaking cannot really be applied,

because some technical conditions for their applicability are not fulfilled. The first one is the condition that the second derivatives of the interaction with respect to the group parameters have to be uniformly bounded over the configuration space of the spins, which fails as a consequence of the noncompact nature of the latter; c.f. (2.2). The second condition is that in the thermodynamic limit we have to have a Gibbs *measure* on the configuration space, whereas in fact our infinite volume state is not a measure, but only a more general *mean* (for more detailed discussion of this see [18]).

The symmetry breaking bc in option (a) of section 2.3 must break the noncompact symmetries and hence amount to something similar to the fixed spin gauge. Also here some potential pitfalls arise, which we illustrate now for the SO(1, 2) model. If one looks at individual configurations of the SO(1, 2) model in the fixed spin gauge at weak coupling (specifically, $\beta = 10$ and $\vec{n}_{x_0=0} = 0$, say), one finds that contrary to the naive expectation expressed above, all spin vectors $\vec{n}_x$ seem to point roughly along the same direction: the system appears to have acquired a large spontaneous magnetization! Writing momentarily $\langle \cdot \rangle_L$ for $\langle \cdot \rangle_{\Lambda,\beta,1}$ with $L_s = L_t =: L$, we find that the 'average magnetization' defined as

$$M = \left(\frac{L^{-4}\langle (\sum_x \vec{n}_x)^2 \rangle_L}{L^{-2}\langle \sum_x \vec{n}_x^2 \rangle_L} \right)^{1/2} , \tag{2.27}$$

does not vanish, rather seems to increase with L: On a 32^2 lattice it is 0.7197, on 64^2 it is 0.7277, and on 128^2, 0.7361 . The reason for this behavior becomes clearer if one considers the 0-component n^0 of the spin: while it is fixed to unity at the origin, it grows logarithmically with the distance. But large zero components necessarily imply large "spatial" components (as $n \cdot n = 1$), and consequently a large ferromagnetic coupling between neighboring spins.

To understand this phenomenon in more detail, it is useful to look at the zero-curvature limit of the SO(1, 2) model, which is just the Gaussian model of a two-component massless free field, defined by the lattice action

$$S = \frac{\beta}{2} \sum_{x,\mu} (\vec{n}_x - \vec{n}_{x+\hat{\mu}})^2 . \tag{2.28}$$

In this model the fixed spin condition means that the spin at the origin $\vec{n}_0$ is fixed to 0. Furthermore, we can compute the counterpart of (2.27) analytically. First one has

$$\langle n_x^i n_y^k \rangle_L = \delta_{ik}[-D(x-y) + D(x) + D(y)] , \tag{2.29}$$

where the well-known function D is given by

$$D(x) = \frac{1}{L^2} {\sum_{l_1,l_2}}' \frac{\exp \frac{2\pi i l \cdot x}{L}}{\sum_\mu (2 - 2\cos \frac{2\pi l_\mu}{L})} , \tag{2.30}$$

with the $\sum'$ running over $l_1, l_2 = 0, 1, \ldots L-1$ but $l_1 = l_2 = 0$ omitted. Using this, we obtain for the square of the numerator of (2.27)

$$\frac{1}{L^4}\Big\langle (\sum_x \vec{n}_x)^2 \Big\rangle_L = \frac{2}{L^2}\sum_x D(x)\,, \tag{2.31}$$

and for the square of the denominator

$$\frac{1}{L^2}\Big\langle \sum_x \vec{n}_x^2 \Big\rangle_L = \frac{4}{L^2}\sum_x D(x)\,. \tag{2.32}$$

It is apparent that in this model one gets $M = 1/\sqrt{2} = .7071$, independent of the lattice size L. The closeness of this number to the numbers quoted above is striking. The small difference between the numbers is due to the curvature of the target space of the $\mathrm{SO}(1,2)$ model, which becomes relevant as the spins fluctuate further from the fixed spin at the origin.This explains why the difference grows with growing L. On the other hand, by increasing β the curvature should become less important; we checked this by measuring M at $\beta = 40$ on a 8^2 lattice and found $M = .707$ in agreement with these expectations.

We conclude that the apparent magnetic ordering of the lattice is due to the fact that the spins fluctuate very far from the origin where $\vec{n}_0 = 0$, and these excursions necessarily take place in a certain direction. If we think of the spins of the noncompact model as the on-mass-shell momentum vectors of a unit mass particle, these vectors are constrained by the action to be such that neighboring particles on the lattice are roughly collinear at weak coupling. In the fixed spin gauge, the fixed vector at the origin then corresponds to a particle at rest, surrounded by "nonrelativistic" neighbors. Far from the origin, the particles become highly relativistic, collinear, and with a local center of momentum frame highly boosted relative to the rest frame of the spin at the origin. This global drift of the center of momentum frame can be prevented by a choice of gauge: indeed, this is precisely the role of the gauge-fixing condition in the translationally invariant gauge, where the spins are described in a frame in which the total spatial momentum vanishes. The fixing of only a single spin is insufficient to arrest the gradual drift of the global center of mass frame in the large volume limit: this phenomenon happens likewise in our model and in the two-component massless free field. Nevertheless one would not ascribe spontaneous symmetry breaking to a Gaussian model.

In fact, this ordering of the lattice is really a gauge artifact: Obviously, in the translation invariant gauge $\sum_x \vec{n}_x$ vanishes and so does the magnetization (2.27). We have checked that by boosting the configurations from the fixed spin gauge to the translation invariant gauge, the magnetization disappears – instead of one dominant direction of the spins we find domains of different spin orientations.

The upshot is that in the discussion of spontaneous symmetry breaking viewpoint (a) should be adopted. The symmetry breaking bc must break the noncompact symmetries

and hence amount to something similar to a gauge fixing of a single spin leaving a maximal compact subgroup, here $\mathrm{SO}^{\uparrow}(N)$, intact. In this gauge we could not find any *local* observable that has a thermodynamic limit and shows breaking of the $\mathrm{SO}^{\uparrow}(N)$ symmetry, and by analogy with the Gaussian model discussed above, we do not think that such an observable exists. The translation invariant gauge fixing is sometimes more convenient but should lead to the same conclusion. In summary, although the usual theorems do *not* apply, we expect that for *local* observables in the two-dimensional model

$$\langle \rho(A)\mathcal{O}\rangle_{\infty,\beta,i} = \langle \mathcal{O}\rangle_{\infty,\beta,i}, \quad i = 1, 2\,, \quad \forall\, A \in \mathrm{SO}^{\uparrow}(N)\,. \tag{2.33}$$

Here ∞ refers to a two-sided thermodynamic limit where $\Lambda \to \mathbb{Z}^2$. We shall offer some further comments on (2.33) in the conclusions.

3. The ground state sector

Spontaneous symmetry breaking, as described above on the level of correlation functions, only precludes the existence of an invariant ground state, without saying much about the set of possible ground states and the possible action of the symmetry group on them. In this section we present a hamiltonian analysis of the ground state sector of the lattice systems in a finite spatial volume, both for discrete and for continuous Euclidean time, that results in a very concrete description of the 'ground state orbit' mentioned in the introduction. The discussion, though limited to systems of finite spatial extent, is valid for $\mathrm{SO}(1,N)$ sigma models in arbitrary (spatial) dimensions. An outlook on the thermodynamic limit via the Osterwalder-Schrader reconstruction is given in Section 3.5.

3.1 Transfer operator in the Schrödinger representation

To address structural issues the transfer operator in the Schrödinger representation is useful. We begin its construction by describing the infinitesimal form of the $\mathrm{SO}(1,N)$ representation ρ in (2.7). Recall that we write $\mathrm{SO}(1,N)$ for $\mathrm{SO}_0(1,N)$. Then $A \in \mathrm{SO(1,N)}$ if and only if A preserves the bilinear form $a\cdot b = a^0 b^0 - a^1 b^1 - \ldots - a^N b^N$ on $\mathbb{R}^{1,N}$ and both sheets of the cone $a \cdot a = 0$, and has unit determinant. In matrix components the first condition becomes ${A_a}^c \eta_{cd} {A_b}^d = \eta_{ab}$, where $\eta = \mathrm{diag}(1,-1,\ldots,-1)$, while the second condition amounts to ${A_0}^0 > 0$. For elements t of the Lie algebra $so(1,N)$ the defining relation reads $ta\cdot b + a \cdot tb = 0$, or ${t_a}^c \eta_{cb} + \eta_{ad} {t_b}^d = 0$. An explicit basis is ${(t^{ab})_c}^d = \delta^a_c \eta^{bd} - \delta^b_c \eta^{ad}$, $0 \leq a < b \leq N$. Consider the 1-parameter subgroups $\mathbb{R} \in s \mapsto \exp\{s t^{ab}\}$, generated by these basis elements and set

$$\rho(t^{ab})\psi(n) = \frac{1}{L_s^d}\frac{d}{ds}\psi\Big(e^{-st^{ab}} n_x,\, x\in\Lambda_t\Big)\Big|_{s=0} = \frac{1}{L_s^d}\sum_{x\in\Lambda_t}\rho_x(t^{ab})\psi(n)\,,$$

$$\text{with} \qquad \rho_x(t^{ab}) = -(t^{ab}n_x)^c \frac{\partial}{\partial n_x^c}\,, \quad x \in \Lambda_t\,. \tag{3.1}$$

The differential operators $\rho_x(t^{ab})$, $0 \leq a < b \leq N$, generate commuting copies of the Lie algebra $\mathrm{so}(1,N)$ at each site:

$$[\rho_x(t^{ab})\,,\, \rho_y(t^{cd})] = \delta_{xy}\,[\eta^{ac}\rho_x(t^{bd}) - \eta^{ad}\rho_x(t^{bc}) + \eta^{bd}\rho_x(t^{ac}) - \eta^{bc}\rho_x(t^{ad})]\,. \tag{3.2}$$

The normalization of $\rho(t^{ab})$ (generating identical infinitesimal transformations in all variables) is adjusted such that they satisfy the same algebra as the t^{ab}. The quadratic Casimir of the local algebras coincides at each site with minus the Laplace-Beltrami operator on $\mathbb{H}_N$ and is given by

$$\mathbf{C}_x = -\Delta_x^{\mathbb{H}_N} = \sum_{a<b,a',b'} \rho_x(t^{ab})\eta_{aa'}\eta_{bb'}\rho_x(t^{a'b'})\,. \tag{3.3}$$

We shall mainly need the following property of $\Delta^{\mathbb{H}_N}$ (omitting the site index momentarily): the spectrum of $-\Delta^{\mathbb{H}_N}$ is purely continuous and is given by the interval $\frac{1}{4}(N-1)^2+\omega^2$, $\omega>0$. Several complete orthonormal systems of improper eigenfunctions are known, see Appendix A and e.g. [25, 26, 27]. In representation theoretical terms this corresponds to a decomposition of the quasi-regular representation on $L^2(\mathbb{H}_N)$ into a direct integral of unitary irreducible representations π_ω of the principal series (where unitarity of the representation refers to an induced inner product; see e.g. [25], Vol.2, Section 10.1.4). For the representation spaces one has

$$L^2(\mathbb{H}_N)=\int^{\oplus} d\omega\,\mu_N(\omega)\,\mathcal{C}_N(\omega)\,, \tag{3.4}$$

with an absolutely continuous spectral measure $\mu_N(\omega)\,d\omega$ given in (A.3) and each irreducible representation occurring with unit multiplicity. Note that the (unitary) singlet representation is not contained in the decomposition (3.4).

Consider now the integral operator T with kernel

$$\begin{aligned} t_\beta(n\cdot n';1) &= D_{\beta,N}^{-1}\exp\{\beta(1-n\cdot n')\}\,,\\ D_{\beta,N} &= 2\Big(\frac{2\pi}{\beta}\Big)^{\frac{N-1}{2}}e^{\beta}K_{\frac{N-1}{2}}(\beta)\,, \end{aligned} \tag{3.5}$$

where $K_\nu(z)$ is a modified Bessel function. The kernel of the iterated operator T^x, $x\in\mathbb{N}$, is denoted by $t_\beta(n\cdot n';x)$. The normalization is such that

$$\int d\Omega(n)\,t_\beta(n\cdot n';x)=1\,,\quad \forall\, n'\in\mathbb{H}_N\,,\ x\in\mathbb{N}\,. \tag{3.6}$$

T can be shown to be a bounded selfadjoint operator on $L^2(\mathbb{H}_N)$ (see [18] for $N=2$). The spectrum of T is absolutely continuous and can be computed exactly (see Appendix A). The spectral values come out as

$$\lambda_{\beta,N}(\omega)=\frac{K_{i\omega}(\beta)}{K_{\frac{N-1}{2}}(\beta)}\,,\quad \omega\geq 0\,. \tag{3.7}$$

They are smooth even functions of ω with a unique maximum at $\omega=0$. Although real and strictly bounded above by $K_0(\beta)/K_{(N-1)/2}(\beta)<1$ they are positive only for $0\leq\omega<\omega_+(\beta)$, where $\omega_+(\beta)$ increases with β like $\omega_+(\beta)\sim\beta+\text{const}\,\beta^{1/3}$. For $\omega>\omega_+(\beta)$ the behavior is oscillatory with exponentially decaying amplitude. Positivity however is restored in the (naive) continuum limit for the Euclidean time: introducing momentarily the lattice spacing a, continuum times $\tau=xa$, as well as a coupling $g^2=1/(\beta a)$, one finds

$$\lim_{a\to 0}[\lambda_{\frac{1}{g^2a},N}(\omega)]^{\frac{\tau}{a}}=\exp\left\{-\tau\frac{g^2}{2}\Big(\frac{(N-1)^2}{4}+\omega^2\Big)\right\}\,. \tag{3.8}$$

This allows one to make contact with the heat kernel, i.e. with the integral kernel $\exp(\frac{\tau g^2}{2}\Delta^{\mathbb{H}_N})(n,n')$ of the exponentiated Laplace-Beltrami operator. From (3.8) one expects

$$\exp\Big(\frac{\tau g^2}{2}\Delta^{\mathbb{H}_N}\Big)(n,n') = \lim_{a\to 0} t_{\frac{1}{ag^2}}\Big(n\cdot n'; \Big[\frac{\tau}{a}\Big]\Big)\,, \tag{3.9}$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}$. On the one hand this can be shown to lead to the correct path integral (see Eq. (II.30) and Appendix D of [26]). On the other hand one can insert the spectral resolution for T to obtain that of the heat kernel. Then the spectral values $\lambda_{\beta,N}$ of T are simply replaced with their continuum counterparts (3.8), see (A.13), (A.15). Massaging this integral further one can show the equivalence to the usual expression for the heat kernel on $\mathbb{H}_N$, quoted in Appendix A. For our purposes a crucial property is the strict positivity

$$\exp\Big(\frac{\tau g^2}{2}\Delta^{\mathbb{H}_N}\Big)(n,n') > 0\,, \qquad \forall\, n,n'\in\mathbb{H}_N\,. \tag{3.10}$$

For finite lattice spacing we can define T in the Schrödinger representation through its spectral resolution. That is,

$$\begin{aligned} T &= \widehat{\lambda}_{\beta,N}(-\Delta^{\mathbb{H}_N})\,, \quad \text{with} \\ \widehat{\lambda}_{\beta,N}(s) &:= \lambda_{\beta,N}\Big(\sqrt{s-(N-1)^2/4}\Big)\,, \quad s\geq \frac{1}{4}(N-1)^2\,. \end{aligned} \tag{3.11}$$

These results for T directly carry over to the 'kinetic' part of the transfer operator $\mathbb{T}_0$ which we define through its integral kernel

$$D_{\beta,N}^{-L_s}\,\exp\Big\{\frac{\beta}{2}\sum_{x\in\Lambda_t}(n_x - n_{x+\hat{2}})^2\Big\} = \prod_{x\in\Lambda_t} t_\beta(n_x\cdot n_{x+\hat{2}};1)\,, \tag{3.12}$$

with the same normalization constant as in (3.5). In the Schrödinger representation this gives

$$\mathbb{T}_0 = \prod_{x\in\Lambda_t}\widehat{\lambda}_{\beta,N}(-\Delta_x^{\mathbb{H}_N})\,. \tag{3.13}$$

In particular $\mathbb{T}_0$ has absolutely continuous spectrum given by

$$\sigma(\mathbb{T}_0) = \sigma_{\rm a.c.}(\mathbb{T}_0) = \overline{\Big\{\prod_{x\in\Lambda_0}\lambda_{\beta,N}(\omega_x)\,\Big|\,\omega_x>0\Big\}}\,, \tag{3.14}$$

where the overbar refers to the closure in $\mathbb{R}$. Its supremum, i.e. the spectral radius $\varrho(\mathbb{T}_0)$ of $\mathbb{T}_0$ equals $\varrho(\mathbb{T}_0) = \lambda_{\beta,N}(0)^{L_s^d}$. On the other hand $\mathbb{T}_0$ is clearly symmetric with respect to the L^2 inner product and it is also bounded in the L^2 norm, since T is. It follows that

$\mathbb{T}_0$ extends to a unique selfadjoint operator on L^2. As such its L^2 norm coincides with its spectral radius: $\|\mathbb{T}_0\| = \varrho(\mathbb{T}_0) = \lambda_{\beta,N}(0)^{L_s^d}$. The improper eigenfunctions of $\mathbb{T}_0$ are of course just direct products of those of T (see Appendix A) and thus are manifestly non-normalizable with respect to the L^2 norm.

To proceed we introduce (unbounded) multiplication operators $\hat{n}_x$, $x \in \{1,\ldots,L_s\}^d$, such that for any function $V(n)$ on $\mathbb{H}_N^{L_s^d}$ (in the Schwartz space of rapidly decreasing smooth functions, say) $V(\hat{n})\psi(n_x,\, x \in \Lambda_t) = V(n_x,\, x \in \Lambda_t)\psi(n_x,\, x \in \Lambda_t)$. Defining the 'potential operator'

$$V(\hat{n}) := \sum_x \sum_{\mu \neq D} (\hat{n}_x \cdot \hat{n}_{x+\hat{\mu}} - 1)\,, \tag{3.15}$$

one readily verifies from (3.12) and (2.4) the following expression for the transfer operator in the Schrödinger representation:

$$\mathbb{T} = \exp\Big\{ -\frac{\beta}{2} V(\hat{n}) \Big\}\; \mathbb{T}_0 \;\exp\Big\{ -\frac{\beta}{2} V(\hat{n}) \Big\}\,. \tag{3.16}$$

It is again bounded and symmetric with respect to the L^2 inner product and hence defines a unique selfadjoint operator. The norm satisfies

$$\|\mathbb{T}\| \leq \|\mathbb{T}_0\| = \lambda_{\beta,N}(0)^{L_s^d} = \left(\frac{K_0(\beta)}{K_{\frac{N-1}{2}}(\beta)} \right)^{L_s^d} < 1\,. \tag{3.17}$$

This means $\mathbb{T}$ is a contraction satisfying $(\psi, \mathbb{T}^t\psi) \leq \|\mathbb{T}\|^t\,(\psi,\psi)$, for all $\psi \in L^2$, $t \in \mathbb{N}$. In particular $\lim_{t\to\infty}(\psi, \mathbb{T}^t\psi) = 0$. The norm $\|\mathbb{T}\|$ is of particular interest because for bounded selfadjoint operator like $\mathbb{T}$ it coincides with the spectral radius and hence can be thought of as "the exponential of minus the ground state energy". In section 3.2 and 3.3 we shall try to narrow in on the associated (improper) eigenspace.

Of course in contrast to $\mathbb{T}_0$ neither the spectrum of $\mathbb{T}$ nor its eigenfunctions are analytically accessible (except, perhaps, for very small L_s). However there are some generic properties which the transfer operator of most lattice systems in finite volume has, but which one should *not* expect $\mathbb{T}$ to have, taking the known properties of $\mathbb{T}_0$ as a guideline. Since one is very much tempted to tacitly assume them, we list the expected 'non-properties' here:

- The eigenfunctions of $\mathbb{T}$ should not be expected to be normalizable with respect to the L^2 norm.
- $\mathbb{T}$ should not be expected to be a positive operator; for discrete Euclidean times then no lattice Hamiltonian, $-\ln\mathbb{T}$, exists.
- The eigenspaces of $\mathbb{T}$ (considered for instance as spaces of smooth bounded functions) carry representations of $\mathrm{SO}(1,N)$ (the restrictions of ρ) which by the first point cannot be expected to be unitary with respect to the L^2 norm.

To the first point one can add that if there was a normalizable ground state it would necessarily have to be noninvariant, by the remark made after Eq. (2.7). The second point is purely technical and could be avoided by working with $\mathbb{T}^2$ instead of $\mathbb{T}$. Alternatively one could start with a different lattice action: the heat kernel action would be an obvious choice. For our purposes the most natural way to ensure the existence of a Hamiltonian is to take the continuum limit in Euclidean time at the outset. This limit exists as we shall argue now.

To this end we assign a coupling β_2 to the kinetic part in $\mathbb{T}$ and a coupling β_1 to the potential part and write $\mathbb{T}_{\beta_1,\beta_2}$ for the result. The k-th power of this operator is given by

$$\mathbb{T}^k_{\beta_1,\beta_2} = e^{-\frac{\beta_1}{2}V}\Big(\mathbb{T}_{0,\beta_2}\, e^{-\beta_1 V}\Big)^k e^{\frac{\beta_1}{2}V}\,, \tag{3.18}$$

in accordance with the integral kernel (2.6). We introduce the lattice spacing a in the Euclidean time direction, writing $\tau = ka$ for the continuum time, and set $\beta_2 = \frac{1}{g^2 a}$, $\beta_1 = \frac{a}{g^2}$. For large β_2 one has $T \sim \exp(\frac{1}{2\beta_2}\Delta^{\mathbb{H}_N})$, so if we heuristically replace T by this heat kernel, we are led to consider instead of (3.18) the sequence

$$e^{-\frac{1}{2kg^2}V}\Big(e^{\frac{\tau g^2}{2k}\sum_x \Delta_x^{\mathbb{H}_N}}\, e^{-\frac{\tau}{kg^2}V}\Big)^k e^{\frac{1}{2kg^2}V}\,, \tag{3.19}$$

which is recognized as Trotter approximant (for $k \to \infty$) of

$$\exp\Big[-\tau\Big(-\frac{g^2}{2}\sum_x \Delta_x^{\mathbb{H}_N} + \frac{1}{g^2}V\Big)\Big]\,, \tag{3.20}$$

where the operator in the exponent is interpreted as the self-adjoint operator given by the form sum of $-\frac{g^2}{2}\sum_x \Delta_x^{\mathbb{H}_N}$ and $\frac{1}{g^2}V$. Since both operators are unbounded but positive, actually Kato's strong Trotter product formula [28] and its refinements [29] could be applied to show that (3.19) converges strongly and even in trace norm to (3.20). But since we used the heat kernel approximation to $\mathbb{T}_0$ above in an informal way, this does not lead to a rigorous proof of the Hamiltonian limit. Probably with more work this could be done, but we do not really need this here; we simply take the semigroup $\mathbb{R}_+ \ni \tau \mapsto \mathbf{T}_\tau$, where

$$\begin{aligned}\mathbf{T}_\tau &= e^{-\tau\mathbb{H}}\,,\\ \mathbb{H} &= -\frac{g^2}{2}\sum_{x\in\Lambda_t}\Delta_x^{\mathbb{H}_N} + \frac{1}{g^2}V\,,\end{aligned} \tag{3.21}$$

as the definition of the continuum dynamics. The essential self-adjointness of $\mathbb{H}$ on the space of smooth functions of compact support can also be seen directly, using the results of [30, 31]. $\mathbf{T}_\tau$ is a strongly continuous contraction semigroup. Both the semigroup

$\mathbf{T}_\tau$, $\tau > 0$, and its generator $\mathbb{H}$ commute with ρ. From (3.21) and (3.17) one infers the bound $\|\mathbf{T}_\tau\| \leq \exp(-\tau L_s^d\, g^2(N-1)^2/8)$, so that

$$\sigma(\mathbb{H}) \geq L_s^d \frac{g^2}{8}(N-1)^2 \,. \tag{3.22}$$

Since $\mathbb{H}$ is an unbounded but manifestly positive operator one could now search for a ground state in the usual way. Technically it is more convenient to work with the bounded positive operator $\mathbb{T}^2$ or the semigroup $\mathbf{T}_\tau$, $\tau > 0$.

3.2 Existence of positive ground state wave functions

We begin by showing that $\mathbb{T}^2$ and $\mathbf{T}_\tau$ have *no normalizable* ground states. Here the concept of a positivity preserving or positivity improving operator T is useful [32]. For convenience we recall the definitions. A nonzero function $\psi \in L^2$ is called positive if $\psi(n) \geq 0$ almost everywhere (a.e.) (that is, outside a set of measure zero in $\mathbb{H}_N^{L_s^d}$ with respect to the product of the invariant measure on $\mathbb{H}_N$) and strictly positive if $\psi(n) > 0$ a.e. Then T is called positivity preserving if $(T\psi)(n) \geq 0$, a.e. and positivity improving if $(T\psi)(n) > 0$, a.e. The latter is equivalent to $(\psi_1, T\psi_2) > 0$ for all positive ψ_1, ψ_2.

The classic use of the positivity improving property is to establish the uniqueness of a ground state once it is known to be normalizable, see [33] or [34] for an application to gauge theories. Here the argument works somewhat differently: the positivity improving property entails that a ground state cannot be normalizable with respect to the L^2 norm.

We first show that both $\mathbb{T}^2$ and the semigroup $\mathbf{T}_\tau$ are indeed positivity improving. For $\mathbb{T}^t$, $t \in \mathbb{N}$, this is obvious because of the strict positivity of the kernels $\mathcal{T}_\beta(n, n'; t)$ in (2.4) and (2.6), rendering all matrix elements with strictly positive functions strictly positive. To show that $\mathbf{T}_\tau$ is positivity improving we use the the path integral representation of this semigroup which is possible because of the strict positivity of the heat kernel (3.10,). From [26, 35, 36] one infers

$$\mathbf{T}_\tau(n, n') = \int \exp\Big(-\int_0^\tau V(\omega(t)dt\Big) d\mu_0(\omega) \;, \tag{3.23}$$

where $d\mu_0(\omega)$ is the measure describing Brownian motion on $\mathbb{H}_N^{L_s^d}$ with starting configuration n and end configuration n' and V is (up to a trivial normalization) the potential defined in (3.15). The paths ω are continuous a.e. with respect to the measure $d\mu_0(\omega)$, and for this reason the integrand is strictly positive a.e., so strict positivity of the kernel $\mathbf{T}_\tau(n, n')$ follows, i.e. $\mathbf{T}_\tau$ is positivity improving. Clearly the $\mathbf{T}_\tau$, $\tau > 0$, are also positive operators while for the transfer operator it convenient to work with the manifestly positive square $\mathbb{T}^2$.

To proceed we recall from (see [32], Section XIII.12) the following general result: if $\|T\|$ is an eigenvalue with a (proper) eigenvector ψ, the eigenvector ψ is nondegenerate and can be chosen strictly positive. The latter statement corresponds to the folklore that a (normalizable) ground state wave function does not have 'nodes'. Applied to the case at hand an immediate consequence of this result is that neither $\mathbb{T}^2$ nor $\mathbf{T}_\tau$ have a normalizable ground state. This is because such a ground state would have to be unique, and thus (since $\mathbb{T}^2$ and $\mathbf{T}_\tau$ commute with ρ) would have to be a singlet under the $\mathrm{SO}(1,N)$ action ρ. However we already know that ρ invariant wave functions are never normalizable because of the 'overcounting' of the group volume, and thus arrive at a contradiction. In other words, for the noncompact sigma-models $\|\mathbb{T}^2\| \notin \sigma_{\rm pp}(\mathbb{T}^2)$, $\|\mathbf{T}_\tau\| \notin \sigma_{\rm pp}(\mathbf{T}_\tau)$, where $\sigma_{\rm pp}$ denotes the pure point spectrum. We are thus faced with the unusual situation that $\|\mathbb{T}^2\|$ and $\|\mathbf{T}_\tau\|$ must lie in the continuous and hence in the essential spectrum of the corresponding operators. In fact [37], $\mathbb{T}^2$ and $\mathbf{T}_\tau$ have *only* essential spectrum

$$\sigma(\mathbb{T}^2) = \sigma_{\rm ess}(\mathbb{T}^2)\;, \quad \sigma(\mathbf{T}_\tau) = \sigma_{\rm ess}(\mathbf{T}_\tau)\,. \tag{3.24}$$

Recall ([38], Section VII.3) that for a bounded selfadjoint operator T the spectrum $\sigma(T)$ decomposes into two disjoint sets, the discrete spectrum $\sigma_{\rm disc}(T)$ and the essential spectrum $\sigma_{\rm ess}(T)$, where $\sigma_{\rm ess}(T)$ is a closed subset of $\mathbb{R}$. In terms of the spectral projectors P_I this amounts to the distinction: $\lambda \in \sigma_{\rm disc}(T)$ iff the range $\mathrm{Ran} P_I$ of P_I is finite dimensional for some open interval I containing λ, and $\lambda \in \sigma_{\rm ess}(T)$ otherwise. Weyl's criterion states that $\lambda \in \sigma(T)$ if and only if there is a family of normalized vectors $(\psi_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|(T-\lambda)\psi_n\| = 0$. Further $\lambda \in \sigma_{\rm ess}(T)$ if and only if the vectors ψ_n can be chosen orthogonal, so that their weak limit vanishes, i.e. $\lim_{n\to\infty}(\psi,\psi_n) = 0$ for all $\psi \in L^2$.

Note that $\mathbb{T}^2$ does also not have ground states in the weak sense, i.e. vectors $\psi_\infty \in L^2$ satisfying $(\psi, (\mathbb{T}^2 - \|\mathbb{T}^2\|)\psi_\infty) = 0$, for all $\psi \in L^2$. This is because such a ψ_∞ would also be a normalizable ground state in the ordinary sense. All this of course applies to $\mathbf{T}_\tau$ as well.

Definitions: In the following we denote by $\mathbf{T}$ a transfer operator without strong or weak ground states. By a *transfer operator* we shall mean a bounded selfadjoint operator that is positive as well as positivity improving (and possibly subject to some subsidary technical conditions). In this situation one will naturally search for weak * ground states of $\mathbf{T}$, i.e. solutions of $(\phi, (\mathbf{T} - \|\mathbf{T}\|)\Omega) = 0$ for all ϕ in a suitable function space, where Ω is a vector in the dual space. Specifically we take $\phi \in L^1$ and call $\Omega \in L^\infty$ a *generalized ground state* of $\mathbf{T}$ if $(\phi, (\mathbf{T} - \|\mathbf{T}\|)\Omega) = 0$ for all $\phi \in L^1$. The set of generalized ground states forms a linear subspace of L^∞ which we call the *ground state sector* $\mathcal{G}(\mathbf{T})$ of $\mathbf{T}$. The existence of generalized ground states which are moreover strictly positive L^∞ functions is guaranteed by a general result [37], a special case of which we describe here.

Let $\mathcal{M}$ be a locally compact space and Ω a regular σ-finite Borel measure on it; let $\mathcal{M} \times \mathcal{M} \ni (m,m') \mapsto \mathcal{T}(m,m') \to \mathbb{R}_+$ be a function that is symmetric, continuous and

strictly positive, i.e. $\mathcal{T}(m,m') > 0$ a.e. We also assume

$$\sup_m \int d\Omega(m')\, \mathcal{T}(m,m') < \infty\,. \tag{3.25}$$

The last condition is sufficient (but by no means necessary) to ensure that $\mathbf{T}$ defines a bounded operator from L^p to L^p for $1 \leq p \leq \infty$; see [39] p.173 ff. The operator norm $\|\mathbf{T}\|_{L^p\to L^p} = \sup_{\|\phi\|_p=1}\|\mathbf{T}\phi\|_p$ is bounded by the integral in (3.25) and coincides with it for $p = 1, \infty$. Positivity of the kernel entails that $\mathbf{T}$ is positivity improving. Positivity of the operator (that is, of its spectrum) is not automatic. However if it is not satisfied we can switch to $\mathbf{T}^2$ and the associated integral kernel, where positivity is manifest. Without much loss of generality we assume therefore the kernel to be such that $\mathbf{T}$ is positive. As a bounded symmetric operator on L^2 the integral operator defined by $\mathcal{T}(m,m')$ has a unique selfadjoint extension which we denote by the same symbol $\mathbf{T}$. The kernel of $\mathbf{T}^L$ will be denoted by $\mathcal{T}(m,m';L)$ for $L \in \mathbb{N}$. In this situation $\mathbf{T}$ and all its powers are transfer operators in the sense of the previous definition.

To prove the existence of generalized ground states as defined above, we need a technical assumption, namely that there exist an $m_* \in M$ (an extremizing configuration) and a subsequence $(L_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\mathcal{T}(m,m;L_j) \leq \mathcal{T}(m_*,m_*;L_j)\,, \quad \forall m \in \mathcal{M} \ \text{ and } \ \forall j \in \mathbb{N}\,. \tag{3.26}$$

The existence of generalized ground states is then guaranteed by

Theorem 1: Let $\mathbf{T}$ be a positive integral operator with kernel $\mathcal{T}(m,m')$ satisfying the conditions listed above. Let $m_* \in \mathcal{M}$ be as in (3.26) and set

$$\Omega_j(v_{m_*}) := \frac{\mathbf{T}^{L_j} v_{m_*}}{(v_{m_*}, \mathbf{T}^{L_j} v_{m_*})}\,, \quad v_{m_*}(m) := \mathcal{T}(m,m_*)\,, \tag{3.27}$$

where $(\ ,\ )$ is the inner product on L^2. Using the assumption (3.26) it is shown in [37] that the L^∞ norms of $\Omega_j(v_{m_*})$ are bounded uniformly in j. Therefore there exists a subsequence $(j_k)_{k\in\mathbb{N}}$ such that the weak * limit

$$\Omega_{m_*} := w^*\!-\lim_{k\to\infty} \Omega_{j_k}(v_{m_*})\,, \tag{3.28}$$

exists; because $(v_{m_*}, \Omega_j(v_{m_*})) = 1$ this limit does not vanish. It is a strictly positive function in L^∞ and a generalized ground state for $\mathbf{T}$, i.e.

$$(\phi, (\mathbf{T} - \|\mathbf{T}\|)\Omega_{m_*}) = 0 \quad \text{for all } \ \phi \in L^1\,. \tag{3.29}$$

Though our present proof requires (3.26) we expect that the conclusion of the Theorem remains valid for a much larger class of transfer operators, which are not necessarily integral operators, and where in particular the condition (3.26) can be dropped.

In the case at hand, all but property (3.26) are manifest for our transfer operator $\mathbb{T}^2$. We conjecture that $\mathbb{T}^2$ satisfies (3.26), in fact with an extremizing configuration $n_* \in \mathbb{H}_N^{L_s^d}$ where all spins equal $n^\uparrow$. In section 3.4 we present in Theorem 2 a stronger result for $\mathbb{T}^2$ which in particular entails the existence of generalized ground states in the above sense.

Either way, the transfer operator $\mathbb{T}^2$ of the $\mathrm{SO}(1,N)$ nonlinear sigma model possesses strictly positive generalized ground states. Based on Theorem 1 (and the conjecture) it comes parameterized by a preferred configuration $n_* \in \mathbb{H}_N^{L_s^d}$, i.e. $\Omega_{n_*} \in L^\infty(\mathbb{H}_N^{L_s^d})$. This then gives rise to an entire orbit $\{\Omega_{An_*}\,,\, A \in \mathrm{SO}(1,N)\}$ of strictly positive generalized ground states. In compact models the counterpart of this orbit would be trivial, i.e. would consist simply of the one-dimensional unitary representation $\Omega_{An_*} = \Omega_{n_*}$, for all $A \in \mathrm{SO}(1{+}N)$. It is a remarkable fact – ultimately rooted in the nonamenability of $\mathrm{SO}(1,N)$ – that this does not happen here.

3.3 The structure of the ground state sector

In fact the ground state sector of the $\mathrm{SO}(1,N)$ nonlinear sigma-models can be described very explicitly. We first summarize the result informally and then give a precise version in the form of a theorem.

$SO(1,N)$ nonlinear sigma-models defined on a finite d-dimensional spatial lattice have infinitely many generalized ground states transforming irreducibly according to the limit of the principal series. Every generalized ground state of the system lies in the linear hull of a single group orbit consisting of strictly positive functions.

Note the sharp contrast to the ground state structure of the compact models:

$SO(N{+}1)$ nonlinear sigma-models defined on a finite d-dimensional spatial lattice have a unique ground state (which is a $SO(N{+}1)$ singlet and which is strictly positive up to a phase).

The precise form of the above statement is:

Theorem 2: Let $\mathbb{T}^2$ be the transfer operator (2.5), (2.6) of the $\mathrm{SO}(1,N)$ nonlinear sigma-model. Then $\mathbb{T}^2$ is a operator on $L^2 = L^2(\mathbb{H}_N^{L_s^d})$ with purely essential spectrum. There *exists* a unique function $\Omega_0(n)$ with the following properties: it is strictly positive and ρ-invariant, i.e. $\rho(A)\Omega_0(n) = \Omega_0(n)$, for all $A \in \mathrm{SO}(1,N)$. Further $\Omega_0(n)$ is independent of one variable, n_{x_0} say, and square integrable in the other variables $\int \prod_{x\neq x_0} d\Omega(n_x)\, \Omega_0(n)^2 < \infty$. In terms of this function and $\mathcal{P}(n) := H_{0,0,0}(n)$ as in (A.10) the ground state sector $\mathcal{G}(\mathbb{T}^2)$ of $\mathbb{T}^2$ is given by

$$\mathcal{G}(\mathbb{T}^2) \simeq \mathrm{Span}\Big\{\Omega_0(n)\,\mathcal{P}(An_{x_0})\,,\;\; A \in \mathrm{SO}(1,N)\Big\}\,. \tag{3.30}$$

In particular all generalized ground states $\Omega \in \mathcal{G}(\mathbb{T}^2) \subset L^\infty$ of $\mathbb{T}^2$ transform according to the limit π_0 of the principal series and are contained in the linear hull of a single group

orbit. Explicitly, the former means they transform equivariantly according to

$$\Omega(A^{-1}n) = (\pi_0(A)\Omega)(n)\,, \quad A \in \mathrm{SO}(1,N)\,. \tag{3.31}$$

The theorem in particular guarantees the existence of generalized ground states in the sense defined in section 3.3. The crucial existence statement is that for the function $\Omega_0(m)$. The proof of Theorem 2 is deferred to [37], where it appears as a special case of more general results. The application to the transfer operator $\mathbb{T}^2$ of the nonlinear sigma-model rests on a technical Lemma which we present here:

Let us denote by $f : \mathbb{H}_N^{L_s^d} \to \mathbb{H}_N^{L_s^d}$ a function satisfying $f_x(n) \cdot f_y(n) = n_x \cdot n_y$ for all $x, y \in \{1, \ldots, L_s\}^d$ and $f_{x_0}(n) = n^\uparrow$ for one x_0. Then

$$\int \prod_x d\Omega(n_x)\, \mathcal{T}_\beta(n, f(n); 2) < \infty\,. \tag{3.32}$$

To see this we express the two-step transfer matrix in terms of the one-step $\mathcal{T}_\beta(n, n'; 1)$ defined in Eq. (2.4). This gives

$$\begin{aligned} \int \prod_x d\Omega(n_x)\, \mathcal{T}_\beta(n, f(n); 2) &= \int \prod_x d\Omega(n_x) d\Omega(n'_x) \\ &\times \exp\Big\{ -\beta \sum_x \sum_{\mu \neq D} [n'_x \cdot (n_x + f_x(n)) + n'_x \cdot n'_{x+\hat{\mu}} + n_x \cdot n_{x+\hat{\mu}} - 4]\Big\}\,. \end{aligned} \tag{3.33}$$

We now estimate $\sum_{x \neq x_0} n'_x \cdot f_x(n) \geq L_s^d - 1$ and view the result of the n_x integrations as a function $F(n')$. It is ρ-invariant so that one of the spins can be frozen to a fixed value, say $n'_{x_0} = n^\uparrow$. The n'_{x_0} integration then can be done and one obtains

$$\begin{aligned} \int \prod_x d\Omega(n_x)\, \mathcal{T}_\beta(n, f(n); 2) &\leq D_{\beta,N}\, e^{-\beta(L_s^d - 1)} \int \prod_x d\Omega(n_x) \prod_{x \neq x_0} d\Omega(n'_x)\, \times \\ &\times \exp\Big\{ -\beta \sum_x \sum_{\mu \neq D} [n_x \cdot n'_x + n'_x \cdot n'_{x+\hat{\mu}} + n_x \cdot n_{x+\hat{\mu}} - 3]\Big\}\Big|_{n'_{x_0} = n^\uparrow}\,. \end{aligned} \tag{3.34}$$

Using $\sum_x n_x \cdot n'_x \geq n_{x_0} \cdot n^\uparrow + L_s^d - 1$ the integrals factorize into products of gauge-fixed one-dimensional partition functions and hence is manifestly finite.

The property (3.32) guarantees that to our transfer operator $\mathbb{T}^2$ (with only essential spectrum) one can associate a compact transfer operator $(\mathbb{T}^2)_0$ (with only discrete spectrum) whose unique ground state wave function is the $\Omega_0(n)$ featuring in Theorem 2; see [37] for details.

As stated in the Theorem, the evolution operators of the nonlinear sigma-models given by $\mathbb{T}^2$ (discrete Euclidean time) or $\mathbf{T}_\tau$ (continuous Euclidean time) both have purely

essential spectrum. The essential spectrum arises from the fact that $\mathbb{T}^2$ or $\mathbf{T}_\tau$ cannot have generalized eigenstates transforming according to a finite dimensional irreducible representation of $\mathrm{SO}(1,N)$. The spectrum of these operators is a closed bounded subset of $[0,\infty)$. Although there can be no normalizable eigenfunctions for the spectral values $\|\mathbb{T}^2\|$ or $\|\mathbf{T}_\tau\|$, it is not excluded that there exist infinite multiplets of normalizable eigenfunctions corresponding to other spectral/eigenvalues. This situation would be interesting in that it might pave the way to a (quasi-)particle interpretation of the spectrum. In contrast to the universal structure of the ground state sector the existence or non-existence of such normalizable multiplets is a specific dynamical feature. In the case of the transfer matrix (3.16) the spectrum remains purely essential for any ρ invariant potential; for certain potentials infinite multiplets of normalizable eigenfunctions might exist. In the absence of a potential, however, this is excluded. Indeed, from (3.14) we know that the kinetic part $\mathbb{T}_0$ and $\exp(-\tau\mathbb{H}_0)$, $\mathbb{H}_0 := -\frac{g^2}{2}\sum_x \Delta_x^{\mathbb{H}_N}$, have absolutely continuous spectrum.

The example of the kinetic part $\mathbb{T}_0$ of the transfer operator (to which all results of course apply in particular) can be used to get a feeling for 'how' the generalized ground states manage to be linear combinations of real and strictly positive functions: a complete system of real eigenfunctions is given by the tensor products of the functions (A.10). Projecting out the $\mathrm{SO}^\uparrow(N)$ singlet yields

$$\begin{aligned}\mathbb{T}_0 H_{\underline{\omega},\underline{L}}(n) &= \prod_{x\in\Lambda_t} \lambda_{\beta,N}(\omega_x)\, H_{\underline{\omega},\underline{L}}(n)\,, \qquad \text{with}\\ H_{\underline{\omega},\underline{L}}(n) &:= \int_{\mathrm{SO}^\uparrow(N)} d\mu(A) \prod_{x\in\Lambda_t} H_{\omega_x,l_x,m_x}(An_x)\,, \end{aligned} \tag{3.35}$$

where the multi-index $\underline{L}$ refers to the set l_x, m_x, $x\in\Lambda_t$, and $\underline{\omega} = (\omega_1,\ldots,\omega_{L_s})$. On the other hand we know from (3.7), (3.14) that the supremum of the spectral values of $\mathbb{T}_0$ is assumed if $\omega_x \to 0$, for all $x\in\Lambda_t$. The limiting eigenfunctions $H_{\underline{0},\underline{L}}$ are real and strictly positive almost everywhere (a.e.). On the other hand an intertwiner $Q(\underline{\omega}|\omega)$ from $\mathcal{C}_N(\omega_1)\otimes\ldots\otimes\mathcal{C}_N(\omega_{L_s})$ to $\mathcal{C}_N(\omega)$ can be seen to contain $\delta(\omega-\sum_x\omega_x)$ as a factor. Assuming that $Q(\underline{\omega}|\omega)$ has a well-behaved limit, the irreducible component $\Omega_\omega(n) = (Q(\underline{\omega}|\omega)H_{\underline{\omega},\underline{L}})(n)$, will likewise be real and positive a.e. as $\omega\to 0$, in accordance with the above result.

Theorem 2 is a special case of far more general results proven in [37]. Roughly speaking the above structure of the ground state sector turns out to arise mainly from the interplay between group theory and the general properties of a transfer operator. It thus admits a generalization largely independent of the details of the dynamics, which we outline here.

Let $\mathbf{T}$ be any transfer operator in the sense defined in section 3.2. Then [37]:

- $\mathbf{T}$ is a noncompact operator on L^2 with purely essential spectrum, $\sigma(\mathbf{T}) = \sigma_{\mathrm{ess}}(\mathbf{T})$.
- Once the existence of a single strictly positive L^∞ ground state is guaranteed the ground state sector $\mathcal{G}(\mathbf{T})$ of $\mathbf{T}$ assumes a certain *universal form*, independent of the details of the dynamics!

- There exists a transfer operator $\mathbf{T}_0$, uniquely associated with $\mathbf{T}$ and with the same spectral radius, such that the ground state sector of $\mathcal{G}(\mathbf{T})$ is related to that of $\mathcal{G}(\mathbf{T}_0)$ by

$$\mathcal{G}(\mathbf{T}) \simeq \mathrm{Span}\Big\{\Omega_0(n)\,\mathcal{P}(An_{x_0})\,, \quad A \in \mathrm{SO}(1,N)\,, \quad \Omega_0 \in \mathcal{G}(\mathbf{T}_0)\Big\}\,. \tag{3.36}$$

- The (generalized) ground states of $\mathbf{T}_0$ are singlets, so that by (3.36) all generalized ground states of $\mathbf{T}$ transform according to the limit π_0 of the principal series: $\Omega(A^{-1}n) = (\pi_0(A)\Omega)(n)\,,\ A \in \mathrm{SO}(1,N)$.

3.4 Thermodynamic limit, SSB, and time-slice bc

In a hamiltonian formulation the thermodynamic limit is hard to control because the 'Hilbert space changes'. The way to proceed is to take the thermodynamic limit on the level of the correlation functions and then reconstruct a Hilbert space formulation via a Osterwalder-Schrader reconstruction. We return to the second aspect in section 3.5. Of course even on the level of expectation values the thermodynamic limit is difficult to control. Interestingly there is an elegant argument saying that the limit of the functional measures (whenever it exists as a mean) cannot be invariant for all $D \geq 1$.

Theorem 3: Expectations $\langle\ \rangle_{\Lambda,\beta,\mathrm{bc}}$ of the $\mathrm{SO}(1,N)$ nonlinear sigma-model defined on a finite D-dimensional lattice Λ with $\mathrm{SO}^{\uparrow}(N)$ invariant bc and gauge fixing cannot have an $\mathrm{SO}(1,N)$ invariant thermodynamic limit $\langle\ \rangle_{\infty,\beta,\mathrm{bc}} := \lim_{\Lambda\to\mathbb{Z}^D}\langle\ \rangle_{\Lambda,\beta,\mathrm{bc}}$. Specifically, there exist bounded continuous functions $\mathcal{O}(n)$ of one spin such that

$$\langle\mathcal{O}(An)\rangle_{\infty,\beta,\mathrm{bc}} \neq \langle\mathcal{O}(n)\rangle_{\infty,\beta,\mathrm{bc}} \quad \text{for some} \quad A \in \mathrm{SO}(1,N)\,. \tag{3.37}$$

As noted in section 2.3 for $D \geq 3$ a similar conclusion was reached in [21] by very different means and based on a different criterion. In the present setting the symmetry breaking is essentially a consequence of the fact that a nonamenable group does not have an invariant mean over 'nice' function spaces [40]. Here we consider the space of bounded continuous functions $\mathcal{C}_b(\mathrm{SO}(1,N))$ on the group manifold. Equipped with the sup-norm it forms a commutative C^*-algebra with unit, so that the usual concept of a state applies.

The proof of Theorem 3 is a straightforward generalization of the argument originally presented for $D = 1$ in [18]. The expectation of a single-spin observable at $x = (x_1, \ldots, x_D)$ can be written as

$$\langle\mathcal{O}(n_x)\rangle_{\Lambda,\beta,\mathrm{bc}} = \int d\mu_{\Lambda,\mathrm{bc}}(n;x)\,\mathcal{O}(n_x)\,. \tag{3.38}$$

By construction, for all finite lattices Λ the one-spin measure $d\mu_{\Lambda,\mathrm{bc}}(n;x)$, $x \in \Lambda_{x_D}$, is a normalized probability measure depending parametrically on L_t. One can thus view (3.38) as bounded, positive, and normalized linear functionals ('states') on the functions in $\mathcal{C}_b(\mathrm{SO}(1,N))$ which happen to be independent of the variables in the $\mathrm{SO}^{\uparrow}(N)$ subgroup. By the theorem of Banach-Alaoglu [38] there is therefore a subsequence of lattices Λ on which the states $\langle\ \ \rangle_{\Lambda,\beta,\mathrm{bc}}$ converge to a limiting state $\langle\ \ \rangle_{\infty,\beta,\mathrm{bc}}$ on $\mathcal{C}_b(\mathrm{SO}(1,N))$. Because $\mathrm{SO}(1,N)$ is non-amenable this limiting state cannot be invariant. There must exist functions $Q \in \mathcal{C}_b(\mathrm{SO}(1,N))$ such that their average in the state $\langle\ \rangle_{\infty,\beta,\mathrm{bc}}$ is noninvariant. For all finite lattices $\langle Q(B)\rangle_{\Lambda,\beta,\mathrm{bc}} = \langle \mathcal{O}(n)\rangle_{\Lambda,\beta,\mathrm{bc}}$ holds, where $\mathcal{O}(n)$, $n \in \mathbb{H}_N = \mathrm{SO}(1,N)/\mathrm{SO}^{\uparrow}(N)$, is the $\mathrm{SO}^{\uparrow}(N)$ average of the function $Q(B)$ and $n = B\,\mathrm{SO}^{\uparrow}(N)$. Since both the observable and the sequence of states are $\mathrm{SO}^{\uparrow}(N)$ invariant, the limit will also be invariant, $\langle Q(B)\rangle_{\infty,\beta,\mathrm{bc}} = \langle \mathcal{O}(n)\rangle_{\infty,\beta,\mathrm{bc}}$. On the other hand by definition of Q one has $\langle \mathcal{O}(An)\rangle_{\infty,\beta,\mathrm{bc}} = \langle Q(AB)\rangle_{\infty,\beta,\mathrm{bc}} \neq \langle Q(B)\rangle_{\infty,\beta,\mathrm{bc}} = \langle \mathcal{O}(n)\rangle_{\infty,\beta,\mathrm{bc}}$, for some $A \in \mathrm{SO}(1,N)$, as claimed.

We remark that when the continuity requirement on the symmetry breaking observable is dropped, the key step in the argument follows more directly from the known characterizations of an amenable symmetric space. A symmetric space G/H (G a locally compact group and H a maximal subgroup) is called *amenable* if there exists a G-invariant mean on $L^{\infty}(G/H)$ (see [41]). A unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called *amenable in the sense of Bekka* if there exists a positive linear functional ω over $\mathcal{B}(\mathcal{H})$ (the C^*-algebra of bounded linear operators on $\mathcal{H}$) such that $\omega(\pi(g)T\pi(g)^{-1}) = \omega(T)$ for all $g \in G$ and all $T \in \mathcal{B}(\mathcal{H})$. Then the following three statements are equivalent: See [42] and [43] (i) G/H is an amenable symmetric space. (ii) the quasiregular representation ρ_1 of G on $L^2(G/H)$ is amenable in the sense of Bekka. (iii) The quasiregular representation ρ_1 almost has invariant vectors in the sense that for all compact $K \subset G$ and all $\epsilon > 0$ there exists a unit vector $\psi \in L^2(G/H)$ such that $\|\rho_1(g)\psi - \psi\| < \epsilon$. Applied to the case at hand we know that the quasiregular representation ρ_1 of $\mathrm{SO}(1,N)$ on $\mathbb{H}_N$ does not almost have invariant vectors, e.g. from the explicit decomposition (3.4). Thus $\mathbb{H}_N$ is a nonamenable symmetric space and by the above argument there must be symmetry breaking observables $\mathcal{O} \in L^{\infty}(\mathbb{H}_N)$, i.e. essentially bounded and measurable functions $\mathcal{O}$ of one spin such that (3.37) holds.

In the rest of this section we present a nonrigorous argument that these symmetry breaking single spin observables are the rule rather than the exception. To this end we introduce expectations with a third type of boundary conditions which take advantage of the extremal configurations $n_* \in \mathbb{H}_N^{L_s^d}$. As stated after Theorem 1 the existence of these extremal configurations, although unproven at present, is highly plausible for the transfer operator $\mathbb{T}^2$ of the noncompact sigma-models. The definition of the expectations $\langle\ \rangle_{\Lambda,\beta,3}$ based on these configurations is as follows. We set

$$\langle \mathcal{O}\rangle_{\Lambda,\beta,3} := \int_{\mathrm{SO}^{\uparrow}(N)} d\mu(A)\, \langle \mathcal{O}\rangle^{An_*}_{\Lambda,\beta,3}\,. \tag{3.39}$$

The expectations referring to n_* are defined for a one-spin observable by

$$\langle \mathcal{O}(n_x)\rangle^{n_*}_{\Lambda,\beta,3} = \int \prod_{x\in\Lambda_{x_D}} d\Omega(n_x)\, \frac{\mathcal{T}_\beta(n_*,n_x;L_t/2+x_D)\mathcal{T}_\beta(n_*,n_x;L_t/2-x_D)}{\mathcal{T}_\beta(n_*,n_*;L_t)}\,, \tag{3.40}$$

for a two-spin observable by

$$\langle \mathcal{O}(n_x,n_y)\rangle^{n_*}_{\Lambda,\beta,3} = \frac{1}{\mathcal{T}_\beta(n_*,n_*;L_t)} \int \prod_{x\in\Lambda_{x_D}} d\Omega(n_x) \prod_{y\in\Lambda_{y_D}} d\Omega(n_y)\,\times \tag{3.41}$$
$$\mathcal{T}_\beta(n_*,n_x;L_t/2+x_D)\,\mathcal{O}(n_x,n_y)\,\mathcal{T}_\beta(n_x,n_y;y_D-x_D)\mathcal{T}_\beta(n_y,n_*;L_t/2-y_D)\,,$$

and so on. The notation in (3.40), (3.41) is the same as in (2.13); compared to the second type of bc the lattice now ranges over time slices Λ_{x_D} , $x_D = -L_t/2,\ldots,0,\ldots,L_t/2$, with L_t even; all spins in the time slices $\Lambda_{\pm L_t/2}$ are frozen to the special configuration n_*. Although we expect n_* to be $\mathrm{SO}^{\uparrow}(N)$ invariant (in that all spins can be chosen to be equal to $n^{\uparrow}$) the bc also work arbitrary $n_* \in \mathbb{H}_N^{L_s^d}$. The invariance under $\mathrm{SO}^{\uparrow}(N)$ (which was a feature of the other two types of bc) then has to be restored by group averaging.

The advantage of these boundary conditions is that by Theorems 1 and 2 the $L_t\to\infty$ limit can be analyzed in a similar way as in the 1-dimensional model [18]. For example for $\mathrm{SO}(1,N)$ invariant observables one has

$$\lim_{L_t\to\infty}\langle \overline{\mathcal{O}}(n_x,n_y)\rangle^{n_*}_{\Lambda,\beta,3} = \|\mathbb{T}\|^{x_D-y_D}\,[2\pi^{N/2}\Gamma(N/2)\Omega_0(n_*)]^{-1} \int \prod_{x\in\Lambda_{x_D}} d\Omega(n_x)\,\times$$
$$\times\overline{\mathcal{O}}(n_*,n_x)\mathcal{T}_\beta(n_*,n_x;y_D-x_D)\Omega_0(n_x)\mathcal{P}((n_*)_{x_0}\cdot n_{x_0})\,. \tag{3.42}$$

Here the limit is taken on a subsequence $(L_t)_{j\in\mathbb{N}}\subset\mathbb{N}$ as in Theorem 1, and $\Omega_0(n)$ is the ρ-invariant positive function in Theorem 2. The normalization is such that the limit functional obeys $\lim_{L_t\to\infty}\langle \mathbb{1}\rangle^{n_*}_{\Lambda,\beta,3} = 1$. This line of argument clearly generalizes to the expectations of observables depending on any finite number of spins. For such observables also the subsequent $L_s\to\infty$ limit exists on subsequences, by the theorem of Banach-Alaoglu. Clearly this argument does not depend on the number of dimensions. We conclude:

The expectations $\langle\overline{\mathcal{O}}\rangle^{n_*}_{\Lambda,\beta,3}$ of all local $SO(1,N)$ invariant observables, defined with $n=n_*$ boundary conditions at $x_D=\pm L_t/2$, have a pointwise finite and explicitly computable thermodynamic limit, $\lim_{L_s\to\infty}\lim_{L_t\to\infty}\langle\overline{\mathcal{O}}\rangle^{n_*}_{\Lambda,\beta,3}$.

For noninvariant observables the evaluation of the thermodynamic limit is more difficult. An exception are observables depending on a single spin only. We shall now argue that basically every nontrivial bounded function of a single spin will signal spontaneous symmetry breaking in the sense of (3.37). This can be seen when using type 3 bc combined with a slightly heuristic use of Theorems 1 and 2. To this end consider the family of measures $d\mu_{\Lambda,3}(n;x)$ in (3.40) with type 3 bc. Let us write $\mathcal{T}_\beta(n_*,n_*;L_t)=$

$O(d(L_t))$ for the leading asymptotics of the denominator. The density of the measures $d\mu_{\Lambda,3}(n;x)$, $x \in \Lambda_{x_D}$, then behaves as

$$d(L_t)\,\Omega_0(n)^2\,\mathcal{P}((n_*)_{x_0}\!\cdot n_{x_0})^2 \quad \text{for} \quad L_t \to \infty\,. \tag{3.43}$$

On general grounds $d(L_t) \to 0$ as $L_t \to \infty$ [37]. The density (3.43) thus vanishes pointwise in the limit. The proof of Theorem 2 is based on the fact that the state space L^2 and the unitary representation ρ can be factorized according to [37]

$$\begin{aligned} L^2 &\simeq L^2(\mathbb{H}_N) \otimes L^2(\mathcal{N}), \quad \text{with} \quad \mathcal{N} = \mathbb{H}_N^{L_s}/\rho(\mathrm{SO}(1,N))\,, \\ \rho &\simeq \rho_1 \otimes \rho_{\mathrm{inv}}^{\Lambda}\,, \end{aligned} \tag{3.44}$$

Here $L^2(\mathbb{H}_N)$ is the factor carrying the dependence on the preferred variable n_{x_0}, and $L^2(\mathcal{N})$ is the factor invariant under the action of ρ. We identify ρ with $\rho_1^{\otimes L_s}$, the L_s-fold tensor product of the quasiregular representation and $\rho_{\mathrm{inv}}^{\Lambda}$ is a representation acting trivially on $L^2(\mathcal{N})$. In particular Ω_0 lies in $L^2(\mathcal{N})$ and thus has finite norm $\|\Omega_0\|_{\mathcal{N}}$ with respect to the invariant part of the measure. Based on this factorization and (3.40), (3.43) one obtains

$$\lim_{L_t\to\infty} \langle \mathcal{O}(n_x)\rangle_{\Lambda,\beta,3} = \|\Omega_0\|^2_{\mathcal{N}} \lim_{L_t\to\infty} \int d\mu_{L_t}(n_{x_0})\,\mathcal{O}(n_{x_0})\,. \tag{3.45}$$

Here $d\mu_{L_t}(n_{x_0})$ is a one-spin measure whose density scales like $d(L_t)\,\mathcal{P}((n_*)_{x_0}\cdot n_{x_0})^2$ for $L_t \to \infty$. The point here is that the second factor on the right hand side of (3.45) is independent of L_s and, whenever the limit $L_t \to \infty$ exists, it is not $\mathrm{SO}(1,N)$ invariant. Assuming that this is the case one can take the $L_s \to \infty$ limit of Eq. (3.45). This affects only the $\|\Omega_0\|^2_{\mathcal{N}}$ term which is ρ invariant for any finite L_s and hence also in the limit. The second factor however is noninvariant and gives rise to spontaneous symmetry breaking – practically for every nontrival bounded one-spin observable, as asserted.

3.5 Osterwalder-Schrader reconstruction

The purpose of the Osterwalder-Schrader reconstruction is to reconstruct a Hilbert space and transfer operator as well as a translation invariant state ('vacuum') Ω from correlation functions (or more generally expectation values) in the thermodynamic limit. Since the infinite volume limit for the transfer matrix cannot be taken ('the Hilbert space changes'), this is the only way in which a physical interpretation of the model in infinite volume can be achieved. The general procedure has been described in many places, see for instance [44, 45, 46]. The following discussion applies to $\mathrm{SO}(1,N)$ sigma-models in any dimension $D \geq 1$. For the 1-dimensional version the Osterwalder-Schrader reconstruction is discussed in detail in [18].

The crucial properties required in a lattice system are

(RP) Reflection positivity, and

(TI) Time translation invariance.

The property of RP in our model can be stated as follows: Denote by $\mathcal{C}_+$ a 'suitable' linear space of continuous functions of finitely many spins n_x at positive 'times'. If $\mathcal{O} \in \mathcal{C}_+$, let $\vartheta\mathcal{O}$ be the complex conjugate of the same function of the time reflected spins. Then

$$\langle \mathcal{O}\vartheta\mathcal{O}\rangle_{\Lambda,\beta,\mathrm{bc}} \geq 0 \ . \tag{3.46}$$

It is satisfied by our model as long as the volume is finite, the temporal size L_t is even and we use boundary conditions that are time-symmetric; this follows from the representation of the system in terms of the transfer matrix $\mathbb{T}$, see Section 2.1. For example it holds in the fixed spin gauge (bc = 2) with periodic bc in time direction if the fixed spin is chosen to have time coordinate $L_t/2$ (identified with $-L_t/2$). It also holds for the fixed time-slice gauge (bc = 3) considered in (3.41), (3.42). In the latter case the existence of a thermodynamic limit is guaranteed at least for $\mathrm{SO}(1,N)$ invariant observables. Translation invariance is manifest already on a finite lattice and thus holds trivially also in the limit. Since for $\mathrm{SO}(1,N)$ invariant functions the different gauge fixes are presumed equivalent this should entail the existence of a thermodynamic limit also for $i = 1, 2$ bc, where the limit is then given by the same formulas as for the type 3 bc. For the fixed spin gauge (i=2) spatially periodic bc are used; translation invariance is not manifest on a finite lattice but should be restored in the limit. We remark that for $\mathrm{SO}(1,N)$ noninvariant observables, such as our 'Tanh' order parameter the properties (RP) and (TI) are not obvious, since in on a finite lattice (RP) is violated in the translation invariant gauge, while (TI) is violated in the fixed spin gauge and in the fixed time slice gauge. Nevertheless it is reasonable to assume that both properties are restored in the thermodynamic limit. For noninvariant observables in $\mathcal{C}_+$ we therefore assume here the existence of a translation invariant thermodynamic limit. We write $\langle\, \cdot \,\rangle$ for the limiting functional $\langle\, \cdot \,\rangle_{\infty,\beta,i}$, $i = 1, 2, 3$, with the above specifications. By construction it then also has the property RP.

The property (RP) allows one to construct a Hilbert space both on a finite lattice and in the thermodynamic limit, which we denote by $\mathcal{H}_\Lambda$ and $\mathcal{H}_{OS}$, respectively. The definitions $(\mathcal{O}', \mathcal{O})_\Lambda := \langle \mathcal{O}\vartheta\mathcal{O}'\rangle_{\Lambda,\beta,\mathrm{bc}}$ and $(\mathcal{O}', \mathcal{O})_{OS} := \langle \mathcal{O}\vartheta\mathcal{O}'\rangle$ define a positive semidefinite scalar product on $\mathcal{C}_+$. By dividing out the subspace of elements with vanishing norm $(\mathcal{O}, \mathcal{O})_{\Lambda,\beta,\mathrm{bc}} = 0$ and $(\mathcal{O}, \mathcal{O})_{OS} = 0$, respectively, and completion we obtain the Hilbert spaces $\mathcal{H}_\Lambda$ and $\mathcal{H}_{OS}$, as described. Importantly $\mathcal{H}_{OS}$ will in general not be separable. This was found explicitly in the 1-dimensional model, and it is unlikely that separability will be restored by 'adding' spatial dimensions. For example the OS reconstruction of the solvable noncompact model of a massless free field in two dimensions likewise leads to a nonseparable state space [47]. In contrast the spaces $\mathcal{H}_\Lambda$ are, for any finite $|\Lambda| = L_s L_t$, isometric to L^2. Denoting the isometry by $V_\Lambda : \mathcal{H}_\Lambda \to L^2$, the unitary representation ρ on L^2 induces one on $\mathcal{H}_\Lambda$, namely $\rho_\Lambda := V_\Lambda^{-1}\rho V_\Lambda$. Our second assumption is that the thermodynamic limit of $\rho_\Lambda(A)$ exists weakly, i.e. the limit $\lim_{|\Lambda|\to\infty}(\rho_\Lambda(A)\mathcal{O}, \mathcal{O}')_\Lambda$

exists for all $\mathcal{O}, \mathcal{O}' \in \mathcal{C}_+$, and defines a measurable function of $A \in \mathrm{SO}(1,N)$. This defines a measurable action ρ_{OS} of $\mathrm{SO}(1,N)$ on $\mathcal{H}_{OS}$. Guided by the properties of the 1-dimensional case, we do *not* expect or require this action to be continuous. Further ρ_{OS} is expected to be unitary only on a closed subspace $\mathcal{H}^u_{OS}$ of $\mathcal{H}_{OS}$. Since $\mathcal{H}_{OS}$ is in general not separable an alternative described by Segal and Kunze applies; see [48] and [18] in the present context. The upshot is that $\mathcal{H}^u_{OS}$ decomposes into a direct sum $\mathcal{H}^u_{OS} = \mathcal{H}^c_{OS} \oplus \mathcal{H}^s_{OS}$, where the restriction of ρ_{OS} to $\mathcal{H}^c_{OS}$ and $\mathcal{H}^s_{OS}$ is continuous and singular, respectively. Here singular means that $(\psi_s, \rho_{OS}(A)\psi_s)_{OS} = 0$ for almost all $A \in \mathrm{SO}(1,N)$ and all $\psi_s \in \mathcal{H}^s_{OS}$. If $\mathcal{H}^u_{OS}$ is separable, $\mathcal{H}^s_{OS}$ is absent. In one dimension such a representation ρ_{OS} could be constructed explicitly, and $\mathcal{H}^s_{OS}$ turned out to be nontrivial. The explicit form of ρ_{OS} also entailed that the Ω induced by $\mathbb{1} \in \mathcal{C}_+$ is actually an element of a ground state orbit $\{\rho_{OS}(A)\Omega,\ A \in \mathrm{SO}(1,N)\}$. The infinite dimensional closed subspace of $\mathcal{H}_{OS}$ spanned by this orbit was contained in $\mathcal{H}^c_{OS}$, that is, the action was continuous and unitary.

The nonamenability of $\mathrm{SO}(1,N)$ now has no direct bearing on the existence of almost invariant vectors for ρ_{OS}, since even a nonamenable group can have amenable representations. However unitary amenable representations π (in the sense of Bekka, defined in the remark following Theorem 3) are characterized by the fact that $\pi \otimes \bar{\pi}$ almost has invariant vectors, where $\bar{\pi}$ is the conjugate representation ([42], Theorem 5.1). We expect that this can be used to rule out that ρ_{OS} restricted to $\mathcal{H}^u_{OS}$ is amenable. Schematically, we identify ρ_{OS} with a weak limit $\rho_{OS} = w - \lim_{|\Lambda|\to\infty} \rho_\Lambda$ (provided it exists) of the unitary representations ρ_Λ on $\mathcal{H}_\Lambda$ as described above. Then, using $\rho_\Lambda \simeq \rho \simeq \rho_1 \otimes \rho^\Lambda_{\mathrm{inv}}$ from (3.44), one formally has

$$\rho_{OS} \otimes \bar{\rho}_{OS} \simeq \rho_1 \otimes \bar{\rho}_1 \otimes \Big(w - \lim_{|\Lambda|\to\infty} \rho^\Lambda_{\mathrm{inv}} \otimes \bar{\rho}^\Lambda_{\mathrm{inv}} \Big) \,. \tag{3.47}$$

Since by assumption ρ_{OS} defines a measurable representation of $\mathrm{SO}(1,N)$ on $\mathcal{H}_{OS}$ and $\rho_1 \otimes \bar{\rho}_1$ does so on $L^2(\mathbb{H}^2_N)$, the second factor in (3.47) should likewise define a measurable representation of $\mathrm{SO}(1,N)$ on some (nonseparable) Hilbert space. Since $\rho_1 \otimes \bar{\rho}_1$ does not have almost invariant vectors it is highly plausible that ρ_{OS} restricted to $\mathcal{H}^u_{OS}$ cannot have almost invariant vectors.

Next we address the reconstruction of a transfer operator. Translation τ by one lattice unit in positive time direction maps $\mathcal{C}_+$ into itself. Since by assumption the limiting expections $\langle\ \ \rangle$ are translation invariant it can be shown by standard arguments (see [44, 45, 46, 18]) that this map lifts to a well-defined bounded symmetric operator $\mathbb{T}_{OS}$ on $\mathcal{H}_{OS}$, which is the desired reconstructed transfer operator. By construction, there is a normalizable state Ω induced by the constant function $\mathbb{1} \in \mathcal{C}_+$, which is a proper eigenstate with eigenvalue unity of $\mathbb{T}_{OS}$ – in sharp contrast to $\mathbb{T}$ which had no proper eigenstates in $L^2 \simeq \mathcal{H}_\Lambda$. We denote by $\mathcal{G}(\mathbb{T}_{OS})$ the set and closed linear subspace of all (normalizable) ground states of $\mathbb{T}_{OS}$ in $\mathcal{H}_{OS}$.

Concering the interplay of $\mathbb{T}_{OS}$ with ρ_{OS} there are two main cases to consider: First the weak limit defining ρ_{OS} does not exist even when restricted to $\mathcal{G}(\mathbb{T}_{OS})$. Second, ρ_{OS} does

exist at least when restricted to $\mathcal{G}(\mathbb{T}_{OS})$, in which case the action of ρ_{OS} can be unitary or nonunitary. In the 1-dimensional model the second possibility was realized with a unitary action. Indeed $\mathbb{T}_{OS}$ commuted with ρ_{OS} on all of $\mathcal{H}_{OS}$. Further $\mathcal{G}(\mathbb{T}_{OS})$ was a subspace of $\mathcal{H}_{OS}^c$ and ρ_{OS} restricted to $\mathcal{G}(\mathbb{T}_{OS})$ was equivalent to π_0, the limit of the principal series. The first possibility should be taken into account based on experience with spontaneous breaking of compact symmetries in dimensions $D \geq 3$. In this case the group no longer acts on the (unique) ground state because of the infinitely many degrees of freedom that would have to be transformed. In the case of a compact symmetry one has the option of averaging the expectation values over the symmetry group, thereby introducing a 'large' (but still separable) Hilbert space as a direct integral over the pure phases. In this Hilbert space there is then degeneracy of the vacuum and the symmetry group acts nontrivially on the vacuum space. The original vacua are recovered by an ergodic decomposition of the symmetric state. Because of the non-amenability of $\mathrm{SO}(1,N)$ one does not have this option here. In summary, we can envisage the following scenarios for the interplay between ρ_{OS} and $\mathbb{T}_{OS}$:

- (1) ρ_{OS} does not exist even when restricted to $\mathcal{G}(\mathbb{T}_{OS})$, or on the restriction $\mathbb{T}_{OS}$ and ρ_{OS} do not commute.
- (2) ρ_{OS} does exist at least when restricted to $\mathcal{G}(\mathbb{T}_{OS})$, and on this subspace $\mathbb{T}_{OS}$ and ρ_{OS} commute. $\mathcal{G}(\mathbb{T}_{OS})$ then decomposes into an orthogonal sum of subspaces, $\mathcal{G}^u(\mathbb{T}_{OS})$ on which ρ_{OS} acts unitarily and $\mathcal{G}^{nu}(\mathbb{T}_{OS})$ on which ρ_{OS} acts nonunitarily. Further ρ_{OS} restricted to $\mathcal{G}^u(\mathbb{T}_{OS})$ is expected to be nonamenable and $\mathcal{G}^u(\mathbb{T}_{OS})$ decomposes in to a direct sum $\mathcal{G}^c(\mathbb{T}_{OS}) \oplus \mathcal{G}^s(\mathbb{T}_{OS})$, where the restriction of ρ_{OS} is continuous on $\mathcal{G}^c(\mathbb{T}_{OS})$ and singular on $\mathcal{G}^s(\mathbb{T}_{OS})$. One or both of these subspaces could be trivial.
- (2a) If $\mathcal{G}^c(\mathbb{T}_{OS})$ is nontrivial it carries a unitary continuous representation of $\mathrm{SO}(1,N)$, which one can assume to be irreducible. Based on the results of Section 3 a plausible candidate is again the limit of the principal series π_0. If $\mathcal{G}^s(\mathbb{T}_{OS})$ is nontrivial the group acts on it discontinuously as a 'permutation group'. Such an exotic situation was found in the $1D$ case for a certain non-vacuum subspace of $\mathcal{H}_{OS}$.

At present we do not have enough information to determine which of the above scenarios holds. All however represent refinements of the fact that the symmetry is spontaneously broken.

4. $D = 2$: Numerical simulations

Although well suited to address structural issues, the hamiltonian formalism used in sections 2 and 3 is not ideal to obtain quantitative results. While in the compact models this is still feasible [49, 50] the intricate group theory required in the noncompact case seems to render such an approach unattractive for models with a noncompact symmetry. In Sections 4 and 5 we therefore study the dynamics in terms of correlations functions, first by numerical simulation and then via the large N expansion.

We have performed simulations of the SO(1,2) sigma-model on square lattices of linear dimension $L = L_s = L_t$ ranging from L=20 to L=128. The simulations were performed at different coupling values, and with the two choices for the gauge-fixing described in Section 2.2. We now describe briefly the Monte Carlo algorithms employed in the simulation of these averages $\langle \mathcal{O} \rangle_{\Lambda,\beta,i}$, $i = 1, 2$:

1. For the average $\langle \mathcal{O} \rangle_{\Lambda,\beta,1}$ (translationally invariant gauge and periodic bc) a Monte Carlo sweep through a $L \times L$ lattice is defined as L^2 Metropolis updates of randomly chosen *pairs* of spins $\vec{n}_{x_1}, \vec{n}_{x_2}$ according to

$$\vec{n}_{x_1} \mapsto \vec{n}_{x_1} + \vec{r}\,, \qquad \vec{n}_{x_2} \mapsto \vec{n}_{x_2} - \vec{r}\,, \tag{4.1}$$

   where the two-dimensional vector $\vec{r} = r(\cos\phi, \sin\phi)$, r, ϕ chosen randomly in the ranges $(0, r_{\max}), (0, 2\pi)$ respectively, and with $r_{\max}$ adjusted to yield acceptance rates close to 50%. The symmetric update of pairs of spins ensures that the gauge constraint $\sum_x \vec{n}_x = 0$ is preserved. The initial configuration (we typically take cold starts, with $\vec{n}_x = 0$) is of course chosen also to satisfy this constraint. The proposed update (4.1) is then accepted or rejected on the basis of the change in the effective action

$$S_1 = \beta \sum_{x,\mu} n_x \cdot n_{x+\hat{\mu}} - 2 \ln \sum_x n_x^0 + \sum_x \ln n_x^0\,. \tag{4.2}$$

   After an initial run of 100000 sweeps to equilibrate the system, configurations are stored subsequently at intervals of 2000 sweeps, which is than adequate to decorrelate the configuration (for the observables measured, typical autocorrelation times are at most a few hundred sweeps). The results presented in this paper derive from averages over ensembles of 5000 independent configurations, unless otherwise stated.

2. For the averages $\langle \mathcal{O} \rangle_{\Lambda,\beta,2}$, a sweep is defined as L^2 updates of a randomly chosen single spin $\vec{n}_{x_1}$, *not including the fixed spin at* x_0,

$$\vec{n}_{x_1} \to \vec{n}_{x_1} + \vec{r}, \quad x_1 \neq x_0\,, \tag{4.3}$$

with the random vector $\vec{r}$ chosen as above, and the effective action in this case

$$S_2 = \beta \sum_{x,\mu} n_x \cdot n_{x+\hat{\mu}} \cdot + \sum_x \ln n_x^0 \tag{4.4}$$

Again, we typically followed an initial equilibration with 100000 sweeps by 5000 measurements spaced at 2000 sweep intervals.

We have found that long simulations of this model lead to unreliable results unless a high quality random number generator is employed. Specifically, violations of translation invariance of two-point functions at the 3 to 4 standard deviation level were found when the random() function packaged with GNU gcc (specifically, gcc-2.96) was used. The ranlux generator developed by M. Lüscher [51], double precision, and with the luxury level set to 2, was used in all simulations reported in this paper. With this generator, we have found no violations outside of statistics in expected symmetries of measured observables.

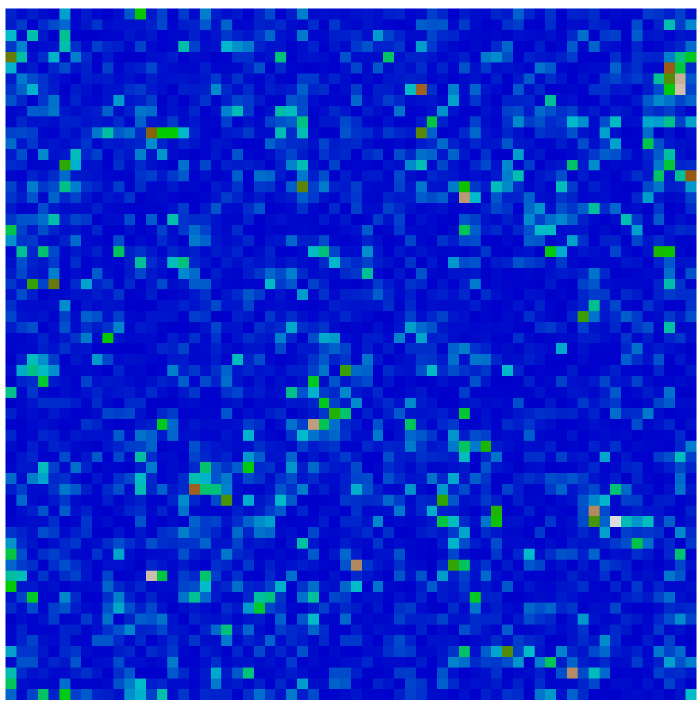


Figure 1: Typical configuration of the 'heights' n_x^0, $x \in \Lambda$, at strong coupling $\beta = 0.1$ for $L = 64$. Blue, green, orange corresponding to low, medium, high values of n_x^0, respectively. The mean value is $\langle n^0 \rangle = 5.11$.

It is instructive to look at some typical configurations in the parametrization $n_x = (\xi_x, \sqrt{\xi_x^2 - 1}\vec{s}_x)$, $x \in \Lambda$. As discussed in Section 2.2 with the translation invariant gauge fixing one expects the $\mathrm{SO}^{\uparrow}(2)$ subgroup to be unbroken. The compact spins $\vec{s}_x$ will then be distributed similar as in the massless phase of the familiar O(2) model. The

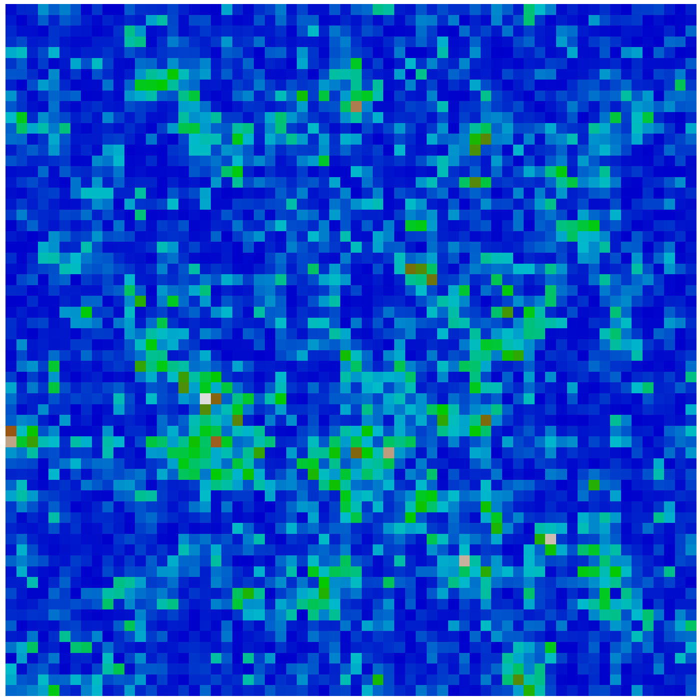


Figure 2: Typical configuration of the 'heights' n^0_x, $x \in \Lambda$, at weak coupling $\beta = 10$ for $L = 64$. Blue, green, orange corresponding to low, medium, high values of n^0_x, respectively. The mean value is $\langle n^0 \rangle = 1.067$.

novel feature are the noncompact components $\xi_x = n^0_x$ for which we show some typical configurations at weak and strong coupling in Figs. 1, 2. One sees that at strong coupling the mean value $\langle n^0 \rangle_{\Lambda,\beta,1}$ is large, with relatively large localized fluctuations rendering nearby spins almost uncorrelated. For weak coupling on the other hand most of the spins are 'frozen' close to bottom of the hyperboloid, $\langle n^0 \rangle_{\Lambda,\beta,1} \sim 1$, and nearby spins are correlated, both in 'height' n^0 and in direction $\vec{n}$.

4.1 Spin two-point function and energy correlator

The spin two-point function $\langle n_x \cdot n_y \rangle_{\Lambda,\beta,i}$ is the simplest $\mathrm{SO}(1,2)$ invariant bilocal object constructible in the model. The thermodynamic limit of this quantity can be studied numerically by simulating various size lattices at fixed β. The results at β =10 for square lattices of linear size $L = \sqrt{|\Lambda|} = 32$, 64, 128 and $i = 1$ (periodic boundary conditions, translationally invariant gauge) are shown in Fig. 3. They suggest the existence of a finite thermodynamic limit, consistent with the analytical arguments in Section 5. It also illustrates that the spin two-point function *increases* with increasing separation $|x - y|$. This somewhat peculiar behavior has been observed before [2] and can be understood analytically both in the 1D model [18] and in a large N analysis, see Section 5.

Another natural invariant observable is the 'energy' or action density $E_x = 2(1 - n_x \cdot$

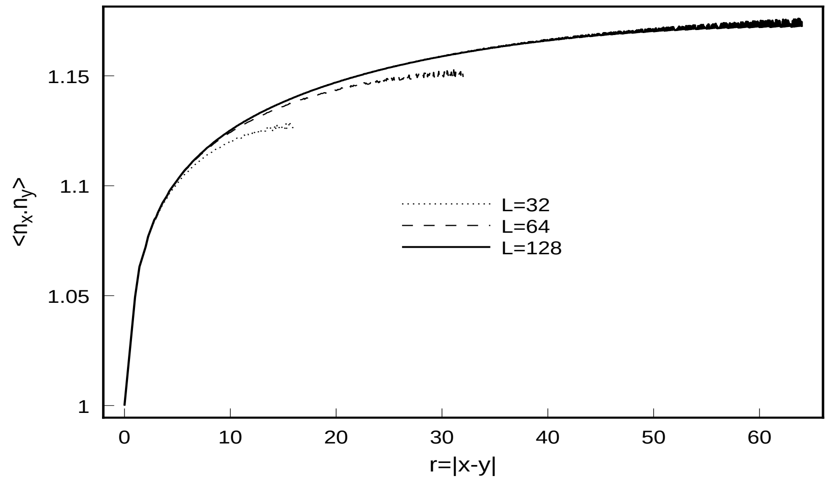


Figure 3: Two-point function $\langle n_x \cdot n_y \rangle_{\Lambda,\beta,1}$, for $\beta = 10$ and varying L.

$n_{x+\hat{\mu}}$). Its expectation value appears in the invariant combination of the Ward identities (2.22). Here we study the connected part of its two-point function and probe for nontriviality and clustering. The subtractions involved in extracting the connected part involve large cancellations, and we have had to perform very long runs (collecting ensembles of 40000 configurations) on somewhat smaller (20×20) lattices to find a meaningful signal. The connected part was also very small in the weak coupling regime, so we needed to go to strong coupling; the results of Fig. 4 correspond to β=0.1. At least for separations $r \leq \sqrt{2}$ lattice spacings there is then a nonvanishing signal. The fact that the signal disappears so rapidly makes it impossible to draw any firm conclusions from the numerical data on the nature of the asymptotic falloff: for example, to distinguish between the r^{-4} power behavior suggested by naive dimensional reasoning, or exponential falloff. In summary, we find a nontrivial energy correlator rapidly decreasing at nonzero separations.

4.2 Two-point function of the Noether current

The Ward identity (2.27) for the longitudinal momentum space current correlators provides a stringent test that the simulation scheme is fully respecting the symmetries of the model. In Fig. 5 we show the comparison of the left and right hand sides of (2.27) on lattices of size 32×32 and 64×64 (periodic boundary conditions, translationally invariant gauge), for the boost ((ab) =(01)) and rotation ((ab) =(12)) Noether currents. The agreement is within statistical errors, except for the lowest momentum modes. In fact, we show in Appendix B that at fixed nonzero momentum the delta function gauge constraint induces a finite volume correction of order $O(\ln V/V)$ to (2.27). These finite volume corrections are largest at the edge of the Brillouin zone, i.e for the momentum modes of order $p \sim 1/L$ (see Appendix B, Fig. 12). The transverse Noether correlators

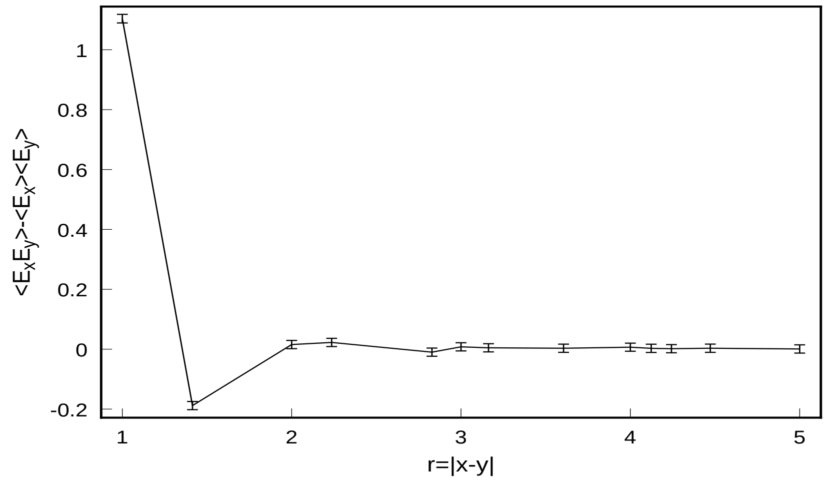


Figure 4: Connected correlator $\langle E_x E_y \rangle - \langle E_x \rangle \langle E_y \rangle$, 20×20 lattice, β=0.1.

are nontrivial, and are shown in coordinate space in Fig. 6. The falloff is roughly $1/r^2$ as expected on dimensional grounds (fits also indicate a logarithmic component $\ln(\mu r)/r^2$).

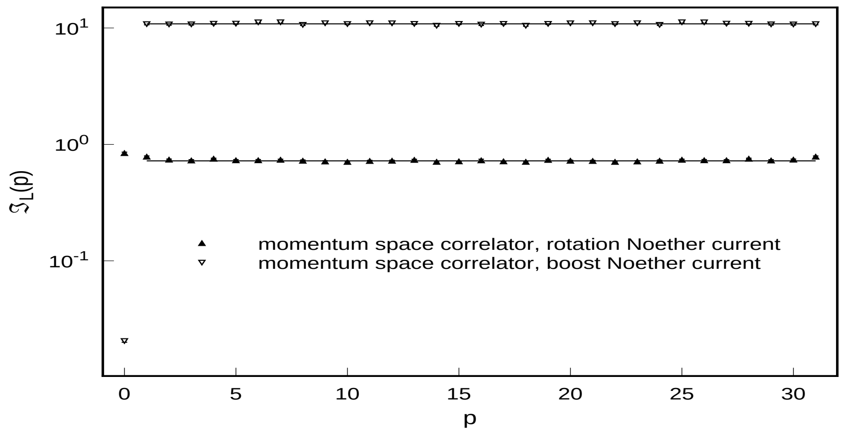
4.3 Tanh observable – spontaneous symmetry breaking

Finally, we have used our generated ensembles to measure the order parameter $T(q)$ introduced in Section 2.4 as a signal of spontaneous symmetry breaking of the SO(2, 1) group. For $N = 2$ the average $\overline{T}_q(\xi)$ in Eq. (2.25) is a strictly decreasing positive function for $\xi \in [1, \infty)$, with the limiting values $\overline{T}_q(1) = \tanh(\sqrt{q^2 - 1})$ and $\overline{T}_q(\infty)$ as below. As described in section 2.3 by a convexity argument one expects

$$\langle\, T_e(n) \rangle_{\Lambda,\beta,1} \geq \overline{T}_q\big(\langle n^0 \rangle_{\Lambda,\beta,1} \big) \geq \overline{T}_q(\infty) = 1 - \frac{2}{\pi} \arccos \sqrt{1 - q^{-2}}\,, \tag{4.5}$$

where the limit is taken from [18]. In Figs. 7, 8, the results for the three functions in (4.5) are shown for weak and strong couplings, respectively. One sees that for weak coupling the spins are almost ‘frozen’ and $\langle\, T_e(n) \rangle_{\Lambda,\beta,1}$ practically coincides with the average $\overline{T}_q\big(\langle n^0 \rangle_{\Lambda,\beta,1} \big)$. For strong coupling, on the other hand, genuine dynamics sets in and both quantities differ. Importantly, in either situation the symmetry breaking is manifest in that $T(q) := \langle T_e(n) \rangle_{\Lambda,\beta,1}$ is a nontrivial function of q. $T_q(1)$ vanishes on account of the unbroken $\mathrm{SO}^{\uparrow}(2)$ symmetry; the lower bound $\overline{T}_q(\infty)$ guarantees that the curve cannot ‘flatten out’ and vanish identically in the thermodynamic limit.

As discussed in Section 2.4 the divergence of $\langle n^0 \rangle_{\Lambda,\beta,1}$ as $|\Lambda| \to \infty$ is not really a test of spontaneous symmetry breaking. Moreover, because of the expected soft (logarithmic) divergence a very large range of lattice sizes would be needed in order to pin-down a suspected divergence of $\langle n^0 \rangle_{\Lambda,\beta,1}$. For example at $\beta = 10$ we obtained $\langle n^0 \rangle_{\Lambda,\beta,1} =$



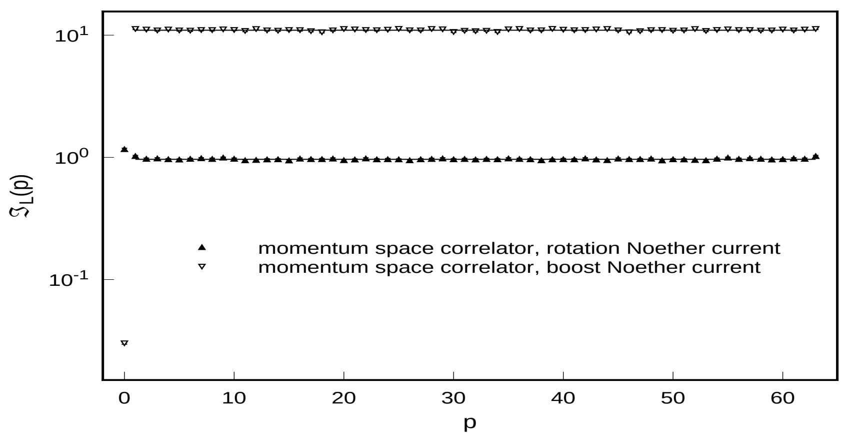


Figure 5: Ward identity for longitudinal Noether currents: $L = 64$, $\beta = 0.1$ (top) and $\beta = 10$ (bottom).

$1.0585, 1.0695, 1.080$, for $L = \sqrt{|\Lambda|} = 32, 64, 128$, respectively. In itself this would hardly constitute convincing evidence for a divergence in the $L \to \infty$ limit.

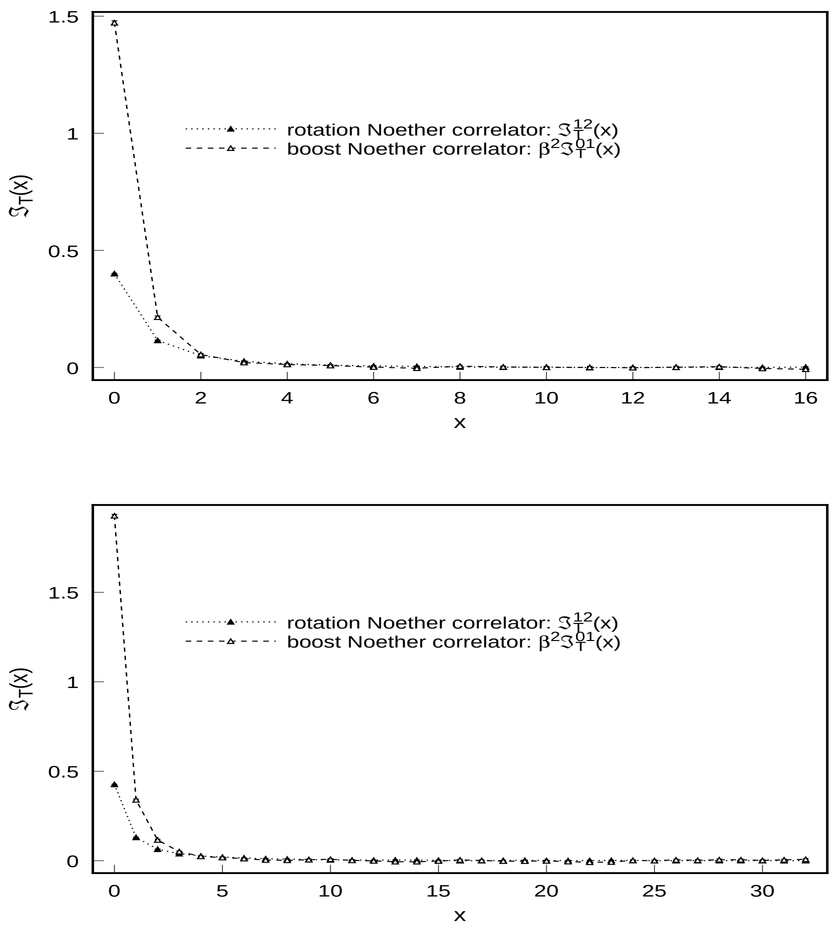


Figure 6: Transverse Noether current correlator, $\beta = 10$, $L = 32,64$.

5. $D = 2$: Large N analysis

The noncompact SO$(1, N)$ sigma-models may be analyzed in the usual large N limit, i.e. $N \to \infty$, $\lambda := N/\beta = g^2 N$ fixed, by saddle-point techniques analogous to those used in the compact case. However, several important differences arise which alter qualitatively the results in the noncompact case. The large N analysis is especially useful for examining qualitative features like the behavior of correlation functions in the thermodynamic limit, thereby providing a guideline for the correct extrapolation of numerical results to the continuum limit. We adopt the setting of Section 2.1 and perform a large N analysis of the lattice-regularized model with periodic boundary conditions, using the translationally invariant gauge-fixing 1 in Section 5.1 and the fixed-spin gauge 2 in Section 5.2. We consider square lattices only and write $\langle \cdot \rangle_{L,N/\lambda,i}$ for $\langle \cdot \rangle_{\Lambda,\beta,i}$, $i = 1, 2$, with $L = \sqrt{|\Lambda|}$ the linear size of the lattice.

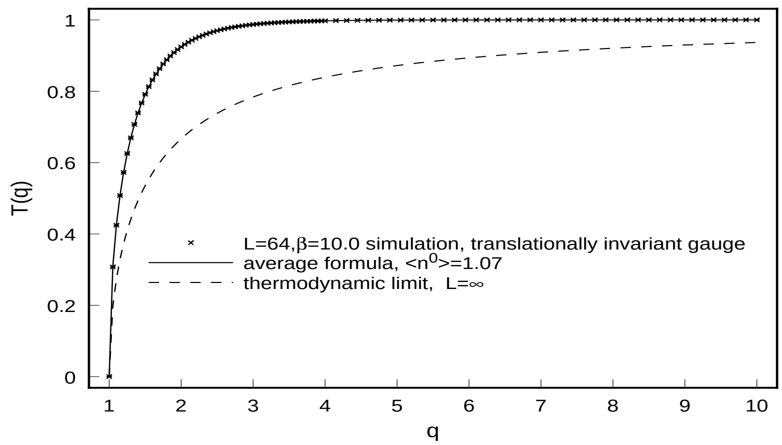


Figure 7: Order parameter $T(q)$ for weak coupling.

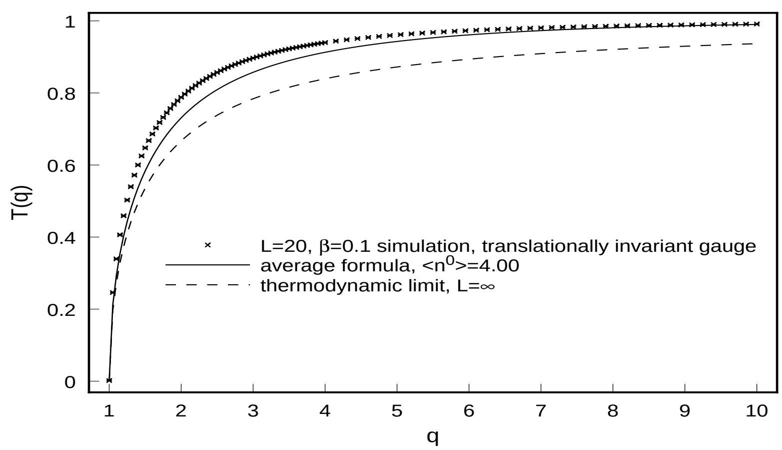


Figure 8: Order parameter T(q) for strong coupling.

5.1 Large N analysis in a translationally invariant gauge

By (2.10) the partition function $Z_1 = Z_1(\Lambda, \beta = N/\lambda)$ has the form

$$\begin{aligned} Z_1 \;&=\; \int \prod_x dn_x \, \delta(n_x^2 - 1) \, \delta\Big(\sum_x \vec{n}_x \Big) \\ &\qquad \times \exp\left\{ N \left(\frac{1}{2\lambda} \sum_{x,\mu} (n^0_{x+\hat{\mu}} - n^0_x)^2 + \ln \Big(\sum_x n^0_x \Big) - \frac{1}{2\lambda} \sum_{x,\mu} (\vec{n}_{x+\hat{\mu}} - \vec{n}_x)^2 \right) \right\}. \end{aligned} \tag{5.1}$$

Implementing the nonlinear constraint as usual with an auxiliary field α_x, (5.1) becomes

$$\begin{aligned} Z_1 \;&=\; \int \prod_x dn^0_x d\alpha_x \exp\left\{ N \left(\frac{1}{2\lambda} \sum_{x,\mu} (n^0_{x+\hat{\mu}} - n^0_x)^2 + \ln \Big(\sum_x n^0_x \Big) - i \sum_x \alpha_x [1 - (n^0_x)^2] \right\} \right. \\ &\times \;\; \int \prod_x d\vec{n}_x \exp\left\{ - N \left(\frac{1}{2\lambda} \sum_{xy} \vec{n}_x (-\Delta)_{xy} \vec{n}_y + i \sum_x \alpha_x \vec{n}_x^2 \right) \right\}, \end{aligned} \tag{5.2}$$

with Δ_{xy} the discrete lattice Laplacian. On integrating out the $\vec{n}$-field one finds, up to irrelevant multiplicative factors

$$Z_1 \sim \int \prod_x dn_x^0 d\alpha_x \exp\left\{-NS_1(n^0,\alpha)\right\}, \tag{5.3}$$

with the effective large N action

$$S_1 = -\frac{1}{2\lambda}\sum_{xy} n_x^0(-\Delta)_{xy}n_y^0 - \ln\sum_x n_x^0 - i\sum_x \alpha_x[(n_x^0)^2-1] + \frac{1}{2}\mathrm{Tr}'\ln\left[-\Delta + 2i\lambda\alpha\right]. \tag{5.4}$$

The prime on the trace in (5.4) denotes omission of the zero mode of the Laplacian, in keeping with the $\delta(\sum_x \vec{n}_x)$ constraint in (4.1).

Note that in contradistinction to the compact case, the n^0 field is not integrated out in defining a large N effective action, and we are led to the problem of determining a joint saddle-point in (n^0,α) field space. We now show that there always exists at least one translationally invariant joint saddle-point of (5.3). Let

$$n_x^0 = \bar{n} + i\eta_x, \qquad \alpha_x = -i\bar{\alpha} + \xi_x\,, \tag{5.5}$$

where $\bar{n},\bar{\alpha}$ are real, but the phase of the fluctuation variables η_x,ξ_x (despite the suggestive notation) remains to be determined later by a detailed analysis of the local structure of the saddle-point. The auxiliary field integrations in (5.3) run initially along the real α_x axes but (as is frequently the case in Hubbard-Stratonovich type saddle-points [52, 53]) require a deformation through a purely imaginary saddle-point. In contrast, the remnant field integrations over n_x^0 run along the semiaxis $[1,\infty)$, and we shall find real saddle-point(s) with $\bar{n}>1$ and real.

The saddle-point conditions $\frac{\partial S}{\partial \eta_x} = \frac{\partial S}{\partial \xi_x} = 0$ yield

$$\bar{n}^2 = 1 + \lambda(-\Delta + 2\lambda\bar{\alpha})^{-1}_{xx}\,, \qquad \bar{\alpha} = -\frac{1}{2V\bar{n}^2}\,, \tag{5.6}$$

where $V := |\Lambda| = L^2$ is the lattice volume. Note that $\bar{\alpha}<0$, corresponding to a *negative* dynamically generated squared mass in the gap equation (5.6). Explicitly this becomes

$$\begin{aligned} f(z) &= 1 - \frac{Vz}{\lambda}\,, \qquad \text{with} \\ f(z) &:= z\sum_{p\neq 0}\frac{1}{2\sum_\mu(1-\cos p_\mu) - z}\,, \qquad z := \frac{\lambda}{V\bar{n}^2}\,. \end{aligned} \tag{5.7}$$

The discrete lattice momenta are $p_\mu = \frac{2\pi}{L}m_\mu$, $m_\mu = 0,1,\dots,L-1$. Due to the infrared divergence of the sum in (5.7) the expectation value of the n^0 field diverges logarithmically for $V\to\infty$, specifically as $\langle (n^0)^2\rangle_{L,N/\lambda,1} \simeq \frac{\lambda}{4\pi}\log V$. Thus the dynamically generated negative squared mass in the gap equation is actually of order $\frac{1}{V\log V}$ in the

N	$\langle n^0 \rangle$
20	3.980 ± 0.007
40	3.808 ± 0.004
80	3.651 ± 0.002
160	3.533 ± 0.002
320	3.458 ± 0.001
640	3.437 ± 0.0005
∞ (saddle-point)	3.4136

Table 1: Comparison of $\langle n^0 \rangle_{L,N/\lambda,1}$ for $L = \lambda = 20$ from simulations with large N result.

thermodynamic limit. Solutions of (5.7) with $f(z) > 0$ correspond to $\bar{n} > 1$. For any V, λ there is always a root with $z < 4\sin\left(\frac{\pi}{L}\right)^2$ and $f(z) > 0$. In the weak coupling regime defined by the inequality $\lambda < 4L^2 \sin\left(\frac{\pi}{L}\right)^2 (\simeq 40$ for large $L)$ it is easy to see that this is the only root yielding $\bar{n} > 1$. Henceforth we shall assume weak coupling, in the sense of the above stated inequality, and dominance of the single saddle-point with $\bar{n} > 1$.

We have performed explicit numerical simulations of the SO$(1, N)$ model at values of N ranging from $N = 20$ up to $N = 640$ on a 20×20 lattice to check the saddle-point result (5.7). The coupling was chosen in the weak coupling regime in the sense indicated above, specifically λ =20. The convergence to the large N limit is shown in Table 1. A fit of the data to a functional dependence of the form $A + B/N + C/N^2$ gives A =3.402. The numerical evidence suggests that the large N functional integral is indeed dominated by the saddle-point located in (5.6).

One may also study field correlators in the large N limit (assuming again the dominance of the saddle-point exhibited above). For example, the SO$(1, N)$ invariant two-point function to leading order in large N takes the simple form

$$\langle n_x \cdot n_y \rangle_{L,N/\lambda,1} \simeq \bar{n}^2 - \lambda D_{xy} \,, \tag{5.8}$$

with $D := [-\Delta + 2\lambda\bar{\alpha}]^{-1}$. Of course $\langle n_x \cdot n_x \rangle_{L,N/\lambda,1} = 1$ as a consequence of the gap equation (5.7). The first term in (5.8) arises from the n^0 correlator, while the second term represents the cumulative effect of N 'spatial' $\vec{n}$ correlators, each of order $1/N$, with a negative sign from the indefinite metric. As the $\vec{n}$ correlators fall off with distance, the SO$(1, N)$ correlator evidently is an increasing function of separation. In Fig. 1 we compare the saddle-point result (5.8) for this correlator with simulation results obtained at finite N on a 20×20 lattice.

Although we shall not compute $1/N$ corrections to the leading large N results here, it is of interest to study the local structure of the saddle-point in this theory, which is a prerequisite to evaluating the Gaussian fluctuation corrections. Indeed, the routing of the field integrations through a joint saddle-point of the type studied here is somewhat more intricate than usual: we shall find that *relative to the implied contour deformations in* (5.5), further contour rotations are required in nonlocally defined field components. Expanding the large N action $S(\bar{n} + i\eta_x, -i\bar{\alpha} + \xi_x)$ to second order in the fluctuation

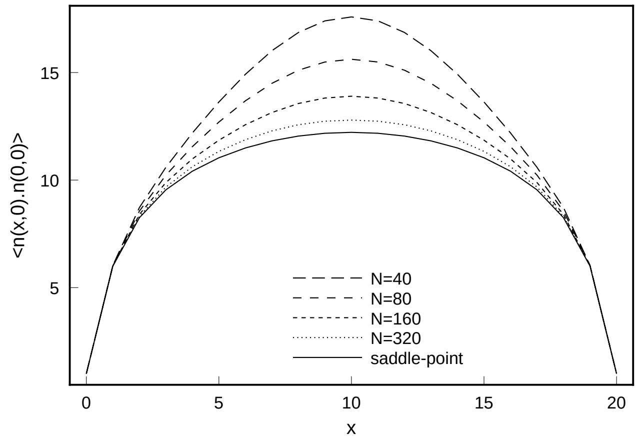


Figure 9: Large N convergence of invariant 2-point function, $L = \lambda = 20$.

fields η_x, ξ_x, one finds the quadratic form

$$S^{(2)} = \frac{1}{2\lambda}\sum_{xy}\eta_x M_{xy}\eta_y - \frac{1}{2\bar{n}^2}\Big(\frac{1}{V}\sum_x \eta_x\Big)^2 + 2\bar{n}\sum_x \eta_x\xi_x + \lambda^2\sum_{xy}(M^{-1})_{xy}\xi_y(M^{-1})_{yx}\xi_x\,, \tag{5.9}$$

with $M := -\Delta + 2\lambda\bar{\alpha}$. The appropriate routing of the field integrations through the saddle (5.5) is best analyzed by going over to momentum space: we replace $\int\prod_x d\eta_x d\xi_x$ by $\int\prod_p d\eta(p)d\xi(p)$ where $\eta_x := \frac{1}{V}\sum_p \eta(p)e^{ip\cdot x}$, etc. The quadratic form (5.9) now becomes

$$S^{(2)} = \frac{1}{V}\sum_p\Big(\frac{1}{2\lambda}M(p)|\eta(p)|^2 + 2\bar{n}\eta(p)\xi(-p)\Big) - \frac{1}{2\bar{n}^2V^2}\eta(0)^2 + \frac{\lambda^2}{V^2}\sum_{p\neq 0,q\neq 0}\frac{|\xi(p-q)|^2}{M(p)M(q)}\,, \tag{5.10}$$

with $M(p) := 4\sum_\mu \sin\left(\frac{p_\mu}{2}\right)^2 + 2\lambda\bar{\alpha}$. We shall henceforth neglect the term involving $\eta(0)^2$ in (5.10), as it is of order $1/V$ relative to the rest. Defining the one-loop polarization function

$$\Pi(p) := \frac{1}{V}\sum_{q\neq 0,r\neq 0}\delta_{p,q-r}\frac{1}{M(q)M(r)}\,, \tag{5.11}$$

the quadratic action can be written as

$$S^{(2)} = \frac{1}{V}\sum_p\Big\{\frac{1}{2\lambda}M(p)|\hat{\eta}(p)|^2 - (\frac{2\lambda}{M(p)}\bar{n}^2 - \lambda^2\Pi(p))|\xi(p)|^2\Big\}\,. \tag{5.12}$$

In (5.12) we change field variables from $(\eta(p), \xi(p))$ to $(\hat{\eta}(p) := \eta(p) + \frac{2\lambda\bar{n}}{M(p)}\xi(p), \xi(p))$. This functional change of variable has unit Jacobian but is of course highly nonlocal in coordinate space. The quadratic action can now be expressed in terms of the variables $\hat{\eta}_R(p) := (\hat{\eta}(p) + \hat{\eta}(-p))/2, \hat{\eta}_I(p) := (\hat{\eta}(p) - \hat{\eta}(-p))/2i$, and similarly for ξ. The integrals over these variables run initially along the real axis, but must be rotated in passing through the saddle-point depending on the sign of $M(p)$ and $\frac{2\lambda}{M(p)}\bar{n}^2 - \lambda^2\Pi(p)$ in (5.12).

For p =0 one has $M(0) < 0, \Pi(0) > 0$. Accordingly the zero mode of the $\hat{\eta}$ field must be rotated by $\pi/2$ in passing through the saddle-point, while the zero mode of the ξ field retains the initial contour orientation. The situation is reversed for nonzero momenta: namely, we find

$$M(p) > 0\,, \qquad \frac{2\lambda}{M(p)}\bar{n}^2 > \lambda^2\Pi(p)\,, \quad \text{for} \;\; p \neq 0\,, \tag{5.13}$$

which implies that for nonzero momentum modes the ξ contours should be rotated by $\pi/2$, while the $\hat{\eta}$ routing is unchanged. Calculations to next to leading order in $1/N$ necessarily require careful attention to the phases induced by these contour rotations. We again emphasize that the above statements hold for the case of a single dominant saddle point, when $\lambda < 4L^2 \sin\left(\frac{\pi}{L}\right)^2$.

5.2 Large N analysis in a fixed-spin gauge

By (2.11) the partition function $Z_2 = Z_2(\Lambda, \beta = N/\lambda)$ reads

$$\begin{aligned} Z_2 \;&=\; \int \prod_x dn_x \delta(n_x^2 - 1)\,\delta(n_{x_0} - n^{\uparrow}) \\ &\quad \times \exp\left\{ N\left(\frac{1}{2\lambda}\sum_{x,\mu}(n^0_{x+\hat{\mu}} - n^0_x)^2 - \frac{1}{2\lambda}\sum_{x,\mu}(\vec{n}_{x+\hat{\mu}} - \vec{n}_x)^2 \right)\right\}. \end{aligned} \tag{5.14}$$

Introducing the auxiliary field α as usual, the partition function becomes in this case

$$Z_2 = \int \prod_{x \neq x_0} dn^0_x d\alpha_x \exp\left\{-NS_2(n^0, \alpha)\right\}, \tag{5.15}$$

with the effective large N action

$$S_2 = -\frac{1}{2\lambda}\sum_{xy} n^0_x(-\Delta)_{xy} n^0_y - i\sum_x \alpha_x[(n^0_x)^2 - 1] + \frac{1}{2}\mathrm{Tr}'\ln\left[-\Delta + 2i\lambda\alpha\right], \tag{5.16}$$

where in (5.16) the prime on the trace now implies a projection corresponding to omission of the integral over the field variable $\vec{n}_{x_0}$. The presence of a fixed spin at a definite point on the lattice now means that the saddle-point will involve (n^0, α) fields with a nontrivial spatial dependence. Writing in analogy to (5.5)

$$n^0_x = \bar{n}_x + i\eta_x\,, \qquad \alpha_x = -i\bar{\alpha}_x + \xi_x\,, \tag{5.17}$$

the saddle-point conditions $\frac{\partial S}{\partial \eta_x} = \frac{\partial S}{\partial \xi_x} = 0$ now lead to

$$\bar{n}^2_x = 1 + \lambda\widehat{D}_{xx}\,, \qquad \bar{\alpha}_x = -\frac{1}{2\lambda}\frac{1}{\bar{n}_x}(\Delta\bar{n})_x\,, \tag{5.18}$$

with the projected propagator $\widehat{D}$ given by

$$\widehat{D}_{xy} = D_{xy} - \frac{D_{x_0x}D_{x_0y}}{D_{x_0x_0}}\,, \qquad D_{xy} = [-\Delta + 2\lambda\bar{\alpha}]^{-1}\,. \tag{5.19}$$

The explicit dependence on a special point x_0 in Eqs. (5.18), (5.19) results in a nontrivial spatial dependence for the solution fields $\bar{n}_x, \bar{\alpha}_x$. For x far from the fixed spin at x_0, we expect the field $\bar{\alpha}_x$ to approach the constant value $\bar{\alpha} = -\frac{1}{2V\bar{n}^2}$ corresponding to the negative dynamically generated squared-mass in the translationally invariant gauge. As $\bar{n}_{x_0}$ is pinned at 1, and $\bar{n}_x > 1$ in general, $\Delta\bar{n}_{x_0} > 0$ and (from (5.18)) $\bar{\alpha}_{x_0} > 0$. So the saddle-point solution in this case involves a spatially dependent dynamical mass, and translational invariance is obviously lost in the propagators $D, \widehat{D}$ above. Nevertheless, if we choose periodic boundary conditions at the edge of the lattice, the fixing of a single spin still only amounts to a SO$(1, N)$ gauge fixing, and SO$(1, N)$ invariant two-point correlators such as

$$\langle n_x \cdot n_y \rangle_{L,N/\lambda,2} \simeq \bar{n}_x\bar{n}_y - \lambda\widehat{D}_{xy}\,, \tag{5.20}$$

must still be translationally invariant, and indeed equal to those found in the translationally invariant gauge, namely (5.8). The equations (5.18), (5.19) cannot be solved analytically, but they are numerically solvable on a given lattice by iteration: one takes a reasonable approximate starting Ansatz for $\bar{n}_x, \bar{\alpha}_x$ and then solves (5.18) for $\bar{n}$ and $\bar{\alpha}$ in alternation until convergence is reached. Single precision convergence (6-7 digits) is typically reached with less than 100 iterations.

A cross-section of the $\bar{n}$ field on a 20×20 lattice (with the fixed spin at the center point $(10, 10)$) is displayed in Fig. 10.

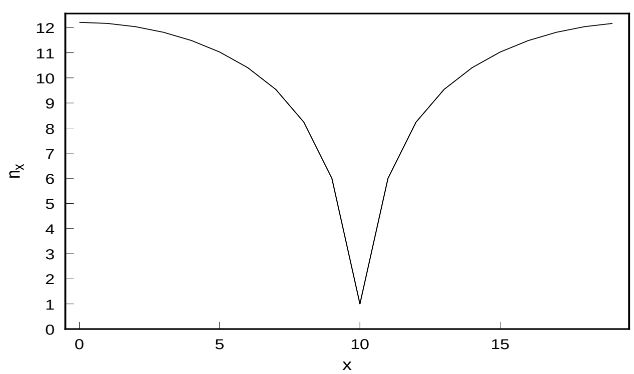


Figure 10: Cross-section of $\bar{n}_x$ for $L = \lambda = 20$.

The figure exhibits the qualitative features discussed above. The $\bar{\alpha}$ field solution in this case (which corresponds to the parameters of Table 1) has a positive spike at the fixed spin with $\alpha_{x_0} = +0.5$, and α_x tending rapidly (within 3 or 4 lattice spacings from the fixed spin in all directions) to the negative constant value $-\frac{1}{2V\bar{n}^2}$ (recall (5.6)) found in the translationally invariant gauge. Note that the typical values of the non-invariant quantity $\bar{n} = \langle n^0 \rangle_{L,N/\lambda,i}$ are very different in the two gauges $i = 1, 2$. Nevertheless, the invariant 2-point functions computed from (5.8) and (5.20) *agree*, as shown in Fig. 11.

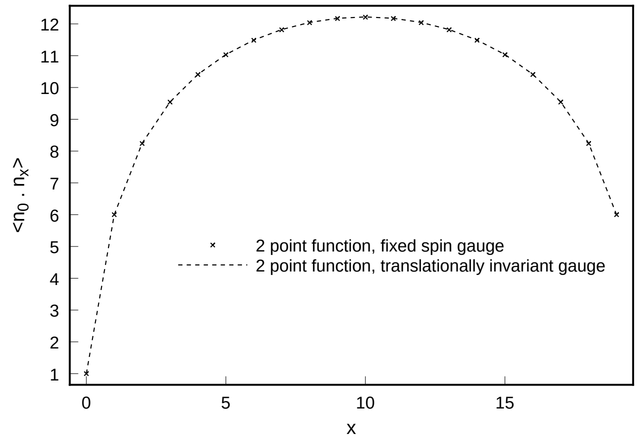


Figure 11: Comparison of large N invariant 2-point correlator in fixed spin and translationally invariant gauges, $L = \lambda = 20$.

6. Conclusions

We have analyzed nonlinear sigma models with noncompact target space and symmetry group $\mathrm{SO}(1,N)$ in dimensions $D \geq 2$ combining analytic and numerical methods. The lattice formulation was used with the dynamics defined both in terms of a transfer operator and a functional integral; in the latter case a gauge fixing was essential. Perhaps the most remarkable feature emerging from the analysis is the intricate vacuum structure as witnessed by Theorem 2. Analyzing the system on a finite spatial lattice via the transfer matrix – where one would usually expect a unique ground state – we could identify a nontrivial ground state orbit, i.e. infinitely many nonnormalizable ground states transforming irreducibly under $\mathrm{SO}(1,N)$. In the thermodynamic limit spontaneous symmetry breaking was found to occur in *all* dimensions $D \geq 1$ (where the case $D = 1$ was already treated in [18]). For dimensions less than three this highlights that the Mermin-Wagner theorem does not hold for these systems. To (numerically) see this spontaneous breakdown on the level of correlation functions, the introduction of a suitable new order parameter ('Tanh') was instrumental. The mathematical reason for these unusual features was understood to be the non-amenability of the symmetry group.

Since in two dimensions the symmetry breaking is surprising we examined this case in more detail. Since the gauge fixing by necessity breaks the symmetry explicitly, it was important to study quantitatively the effect of this explicit breaking via Ward identities. Our numerical simulations show clearly that this explicit violation disappears in the thermodynamic limit, whereas the symmetry breaking shown by the 'Tanh' order parameter remains. The new order parameter thus provides for noncompact models a numerically effective way to probe for 'spontaneous' symmetry breaking in finite volume.

In addition we performed a large N saddle point analysis in the two-dimensional models. The qualitative features we deduced for the model at finite N were confirmed explicitly

in the solution of the $N = \infty$ model.

A variety of open questions remain: what is the significance of the spontaneous symmetry breaking for the localization phenomena the two-dimensional sigma-models provide an effective description of? Further are these systems integrable and amenable to a bootstrap construction, e.g. based on an R-matrix with the symmetry of the principal unitary series representations like in [54]? For numerical simulations a (hybrid-) cluster algorithm would be desirable; in particular to probe whether or not the peculiar long range order (sensitivity to boundary conditions) found in the 1-dimensional model extends to the field theories. An important question is of course whether there exists a nontrivial continuum limit. Conventional wisdom would say no, because the models are (perturbatively) asymptotically free in the infrared but not in the ultraviolet ([55, 10]). To gain some feeling whether this is true beyond perturbation theory, it might be worthwhile to study simplified hierarchical versions of the Renormalization Group. Finally the vacuum structure in the reconstructed Hilbert space should investigated further, as well as its relevance for dimensionally reduced gravity theories, where a continuum limit is expected to exist.

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Appendix A: Spectral decompositions and heat kernel on $\mathbb{H}_N$

Let $-\Delta^{\mathbb{H}_N}$ be minus the Laplace-Beltrami operator on the hyperboloid $\mathbb{H}_N$, $N \geq 2$. Recall that its spectrum is absolutely continuous and is given by the interval $\frac{1}{4}(N-1)^2 + \omega^2$, $\omega > 0$. There are several complete orthogonal systems of improper eigenfunctions. From a group theoretical viewpoint the most convenient system are the 'principal plane waves' $E_{\omega,k}(n)$ (see [56, 25, 27] and the references therein) labeled by $\omega > 0$ and a 'momentum' vector $\vec{k} \in S^{N-1}$. Parameterizing $n = (\xi, \sqrt{\xi^2 - 1}\,\vec{s})$, they read

$$E_{\omega,k}(n) = [\xi - \sqrt{\xi^2 - 1}\,\vec{s} \cdot \vec{k}]^{-\frac{1}{2}(N-1) - i\omega} . \tag{A.1}$$

The completeness and orthogonality relations take the form

$$\begin{aligned} \int d\Omega(n)\, E_{\omega,k}(n)^* E_{\omega',k'}(n) &= \frac{1}{\mu_N(\omega)} \delta(\omega - \omega')\delta(\vec{k}, \vec{k}') , \\ \int_0^\infty d\omega\, \mu_N(\omega) \int_{S^{N-1}} dS(k) E_{\omega,k}(n)^* E_{\omega,k}(n') &= \delta(n, n') , \end{aligned} \tag{A.2}$$

where $\delta(n, n')$ and $\delta(\vec{k}, \vec{k}')$ are the normalized delta distributions with respect to the invariant measures $d\Omega(n)$ and $dS(k)$ on $\mathbb{H}_N$ and S^{N-1}, respectively. The spectral weight is

$$\mu_N(\omega) = \frac{1}{(2\pi)^N} \left| \frac{\Gamma(\frac{N-1}{2} + i\omega)}{\Gamma(i\omega)} \right|^2 . \tag{A.3}$$

The main virtue of these functions is their simple transformation law under $\mathrm{SO}(1, N)$. For $A \in \mathrm{SO}^{\uparrow}(N)$ one has trivially $E_{\omega,k}(A^{-1}n) = E_{\omega,Ak}(n)$. To describe the action of the boosts let $\vec{a} \in S^{N-1}$ and decompose $\vec{n} = \sqrt{\xi^2 - 1}\,\vec{s}$ into its components parallel $\vec{n}_\parallel$ and orthogonal $\vec{n}_\perp$ to $\vec{a}$. A boost in the direction $\vec{a}$ will then leave $\vec{n}_\perp$ invariant. Denoting the boost parameter by $\theta \in \mathbb{R}$ the corresponding element $A = A(\theta, a) = A(-\theta, a)^{-1}$ acts by

$$A(\theta, a)^{-1} \begin{pmatrix} \xi \\ \vec{n} \end{pmatrix} = \begin{pmatrix} \xi \mathrm{ch}\theta - \mathrm{sh}\theta\, \vec{n} \cdot \vec{a} \\ \mathrm{ch}\theta\, \vec{n}_\parallel - \mathrm{sh}\theta\, \xi \vec{a} + \vec{n}_\perp \end{pmatrix} . \tag{A.4}$$

Using this in (A.1) one verifies

$$E_{\omega,k}(A^{-1}n) = [\mathrm{ch}\theta + \vec{a} \cdot \vec{k}\, \mathrm{sh}\theta]^{-\frac{1}{2}(N-1) - i\omega} E_{\omega, r_A(k)}(n) , \tag{A.5}$$

where $\vec{r}_A(\vec{k}) \in S^{N-1}$ is a rotated momentum vector whose components $\vec{r}_A(\vec{k})_\parallel$ parallel and orthogonal $\vec{r}_A(\vec{k})_\perp$ to $\vec{a}$ are given by

$$\begin{aligned} \vec{r}_A(\vec{k})_\parallel &= \frac{\vec{a} \cdot \vec{k}\, \mathrm{ch}\theta + \mathrm{sh}\theta}{\mathrm{ch}\theta + \vec{k} \cdot \vec{a}\, \mathrm{sh}\theta}\, \vec{a} , \\ \vec{r}_A(\vec{k})_\perp &= \frac{\vec{k} - (\vec{a} \cdot \vec{k})\vec{a}}{\mathrm{ch}\theta + \vec{k} \cdot \vec{a}\, \mathrm{sh}\theta} . \end{aligned} \tag{A.6}$$

The transformation law (A.5) characterizes the (even parity) principal unitary series $\mathcal{C}_N(\omega)$, $\omega > 0$ of SO$(1,N)$, where $\mathcal{C}_N(\omega)$ and its complex conjugate are unitary equivalent (see e.g. [25],Vol.2, Sections 9.2.1 and 9.2.7). The orthogonality and completeness relations (A.2) amount to the decomposition (3.4) of the quasi-regular representation ρ on $L^2(\mathbb{H}_N)$. Further one verifies

$$dS(k) = [\mathrm{ch}\theta + \vec{a}\cdot\vec{k}\,\mathrm{sh}\theta]^{N-1} dS(r_A(k))\,. \tag{A.7}$$

This implies that $dS(k)$ integrals over products of the form $E_{\omega,k}(n)^* E_{\omega,k}(n')$ are invariant under the SO$(1,N)$ action (A.5). In particular one can define spectral projectors P_I commuting with ρ in terms of their kernels $P_I(n\cdot n')$, $I \subset \mathbb{R}_+$:

$$\begin{aligned} P_I(n\cdot n') &:= \int_I d\omega \mu_N(\omega) \int_{S^{N-1}} dS(k)\, E_{\omega,k}(n)^* E_{\omega,k}(n') \\ \int d\Omega(n') P_I(n\cdot n') P_J(n'\cdot n'') &= P_{I\cap J}(n\cdot n'')\,. \end{aligned} \tag{A.8}$$

Combined with the completeness relation in (A.2) this shows that the spectra of $-\Delta^{\mathbb{H}_N}$ and of T in (3.5) are absolutely continuous.

A complete orthogonal set of real eigenfunctions of $-\Delta_{\mathbb{H}_N}$ is obtained by taking the $dS(k)$ average of the product of $E_{\omega,k}(n)$ with some spherical harmonics on the k-sphere. This amounts to a decomposition in terms of SO$^{\uparrow}(N)$ irreps where the 'radial' parts of the resulting eigenfunctions are given by Legendre functions. Using the normalization and the integral representation from ([57] p. 1000) one has in particular

$$\int_{S^{N-1}} dS(k)\, E_{\omega,k}(n) = (2\pi)^{N/2} (\xi^2-1)^{\frac{1}{4}(2-N)}\, \mathcal{P}^{1-N/2}_{-1/2+i\omega}(\xi)\,. \tag{A.9}$$

As a check on the normalizations one can take the $\xi \to 1^+$ limit in (A.9). The limit on the rhs is regular and gives $2\pi^{N/2}/\Gamma(N/2)$, which equals the area of S^{N-1} as required by the limit of the lhs. Denoting the set of real scalar spherical harmonics by $Y_{l,m}(k)$, $l \in \mathbb{N}_0$, $m = 0,\ldots,d(l)-1$, with $d(l) = (2l+N-2)(l+N-3)!/(l!(N-2)!)$ we set

$$H_{\omega,l,m}(n) := \int dS(k)\, Y_{l,m}(k) E_{\omega,k}(n) \tag{A.10a}$$

$$= k_l(\omega)\, Y_{l,m}(\vec{s})\, (\xi^2-1)^{\frac{1}{4}(2-N)}\, \mathcal{P}^{1-N/2-l}_{-1/2+i\omega}(\xi)\,, \quad \text{with} \tag{A.10b}$$

$$k_0(\omega) = (2\pi)^{N/2}\,, \quad k_l(\omega) = (2\pi)^{N/2} \left(\prod_{j=0}^{l-1} \Big[\omega^2 + (\tfrac{N-1}{2}+j)^2\Big] \right)^{1/2}, \; l \geq 1\,.$$

The expression (A.10b) is manifestly real, the equivalence to (A.10a) can be seen as follows: from (A.2), (2.3), and the orthogonality and completeness of the spherical

harmonics one readily verifies that both (A.10a) and (A.10b) satisfy

$$\begin{aligned}\int d\Omega(n)\, H_{\omega,l,m}(n)^* H_{\omega',l',m'}(n) &= \frac{1}{\mu_N(\omega)}\delta(\omega-\omega')\delta_{l,l'}\delta_{m,m'}\,,\\ \int_0^\infty d\omega\, \mu_N(\omega) \sum_{l,m} H_{\omega,l,m}(n)^* H_{\omega,l,m}(n) &= \delta(n,n')\,. \end{aligned}\tag{A.11}$$

Further both (A.10a) and (A.10b) transform irreducibly with respect to the real $d(l)$ dimensional matrix representation of $\mathrm{SO}^\uparrow(N)$ carried by the spherical harmonics. Hence they must coincide. A drawback of the functions (A.10) is that the $\vec{k}$ integration spoils the simple transformation law (A.5) under $\mathrm{SO}(1,N)$. The transformation law can now be inferred from the addition theorem

$$\sum_{l,m} H_{\omega,l,m}(n) H_{\omega,l,m}(n') = (2\pi)^{N/2}\,[(n\cdot n')^2-1]^{\frac{1}{4}(2-N)}\,\mathcal{P}^{1-N/2}_{-1/2+i\omega}(n\cdot n')\,. \tag{A.12}$$

For example for $n' = An^\uparrow$ this describes the transformation of the $\mathrm{SO}^\uparrow(N)$ singlet $H_{\omega,0,0}(n)$ under $A \in \mathrm{SO}(1,N)$.

Having laid out the relevant representation theory let us consider the spectral decomposition of the various operators under consideration. For the kernel (3.5) of T we use an ansatz of the form

$$t_\beta(n\cdot n';1) = \int_0^\infty d\omega \mu_N(\omega)\, \lambda_{\beta,N}(\omega) \int_{S^{N-1}} dS(k)\, E_{\omega,k}(n)^* E_{\omega,k}(n')\,, \tag{A.13}$$

chosen such that $TE_{\omega,k} = \lambda_{\beta,N}(\omega)E_{\omega,k}$. To determine the eigenvalues we set $n' = n^\uparrow$ and integrate over k. Using (2.3), (A.9), and the integral ([57], p.804) one finds

$$\lambda_{\beta,N}(\omega) = (2\pi)^{N/2} e^\beta D^{-1}_{\beta,N} \int_1^\infty d\xi (\xi^2-1)^{\frac{1}{4}(N-2)} e^{-\beta\xi} \mathcal{P}^{1-N/2}_{-1/2+i\omega}(\xi) = \frac{K_{i\omega}(\beta)}{K_{\frac{N-1}{2}}(\beta)}\,, \tag{A.14}$$

as asserted in (3.7). The spectral representation of the iterated kernel $t_\beta(n\cdot n';x)$, $x\in\mathbb{N}$, equals (A.13) just with $\lambda_{\beta,N}(\omega)$ replaced by $[\lambda_{\beta,N}(\omega)]^x$.

In view of (3.9), (3.8) this directly yields an integral representation for the heat kernel which (after rescaling $\tau g^2/2 \to \tau$) reads:

$$\exp(\tau\Delta^{\mathbb{H}_N})(n,n') = \int_0^\infty d\omega\, \mu_N(\omega)\, e^{-\tau[(\frac{N-1}{2})^2+\omega^2]} \int_{S^{N-1}} dS(k)\, E_{\omega,k}(n)^* E_{\omega,k}(n')\,. \tag{A.15}$$

Let us briefly recap the main properties of the heat kernel on $\mathbb{H}_N$, see e.g. [58]

(i) $\exp(\tau\Delta^{\mathbb{H}_N})(n,n')$ is symmetric in n,n' and is a bi-solution of the heat equation $\frac{\partial}{\partial\tau}u = \Delta^{\mathbb{H}_N}u$.

(ii) for each $n' \in H_N$, $d\Omega(n)\exp(\tau\Delta^{\mathbb{H}_N})(n,n')$ is a probability measure which converges to the Dirac measure $\delta(n,n')$ as $\tau \to 0^+$.

(iii) it is invariant $\exp(\tau\Delta^{\mathbb{H}_N})(An,An') = \exp(\tau\Delta^{\mathbb{H}_N})(n,n')$, $A \in \mathrm{SO}(1,N)$, and hence a function of $r = \mathrm{arccosh}(n\cdot n')$ only, for which we write $h_\tau(r)$.

(iv) $h_\tau(r)$ is smooth and strictly positive for all $r \geq 0$ and $\tau > 0$; in particular the coincidence limit $r \to 0^+$ is finite.

Most of these properties are readily verified from the spectral representation (A.15). Properties (i) and (iii) are manifest. The limit $\lim_{\tau\to 0}\exp(\tau\Delta^{\mathbb{H}_N})(n,n') = \delta(n,n')$ follows from (A.2). The fact that $\int d\Omega(n')\exp(\tau\Delta^{\mathbb{H}_N})(n,n') = 1$ for all $n \in \mathbb{H}_N$ and $\tau > 0$, is a consequence of (3.9) and (3.6). This gives (ii). The finiteness of the coincidence limit is clear from (A.9) and the remark after it. However the positivity in (iv) is masked by the oscillating nature of the Legendre functions. It can be shown, for example, from the alternative expressions (A.16), (A.17) below, where also the smoothness is manifest.

Using the behavior of the Legendre $\mathcal{P}^\mu_\nu$ functions under a sign flip of μ one can rewrite (A.15) in a form where a simplified spectral weight appears which equals that of the $N = 1$ case for all odd N and that of $N = 2$ case for all even N. With some further processing one can show [26] the equivalence to the usual expressions for the heat kernel. For $N = 2$ see e.g. [59] Vol. 1, Eqs (3.32), (3.33). For $N > 2$ see [26]. The final result we quote from [58]

$$h_\tau(r) = \sqrt{\pi}(2\pi)^{-\frac{N+1}{2}}\tau^{-1/2}e^{-\frac{(N-1)^2}{4}\tau}\Big(-\frac{1}{\mathrm{sh}r}\frac{\partial}{\partial r}\Big)^{\frac{N-1}{2}} e^{-\frac{r^2}{4\tau}}\,, \quad N\,\text{odd} \tag{A.16}$$

$$h_\tau(r) = (2\pi)^{-\frac{N+1}{2}}\tau^{-1/2}e^{-\frac{(N-1)^2}{4}\tau}\int_r^\infty \frac{ds\,\mathrm{sh}s}{\sqrt{\mathrm{ch}s - \mathrm{ch}r}}\Big(-\frac{1}{\mathrm{sh}s}\frac{\partial}{\partial s}\Big)^{\frac{N}{2}} e^{-\frac{s^2}{4\tau}}\,, \quad N\,\text{even}\,. \tag{A.17}$$

From here one readily verifies the positivity property in (iv).

Appendix B: Finite volume corrections to Ward identities

Here we derive for $D = 2$ the finite volume corrections to the Ward identity (2.22), (2.23). We use the translation invariant gauge fixing 1 of Section 2.1 where the $\mathrm{SO}(1,N)$ invariance is violated by delta function constraint in Eq. (2.10). One may expect such corrections to vanish in the thermodynamic limit. In fact, it is possible to calculate the explicit form of these corrections and thereby study directly their volume dependence. The most convenient approach starts with a derivation of the exact Ward identity in a nonsingular gauge analogous to the λ (or ξ)-gauges of quantized nonabelian gauge theories, and then recovers the delta-function gauge by taking the limit $\lambda \to \infty$. Thus, we begin with the functional integral

$$Z_\lambda \;=\; \int \prod_x d\vec{n}_x \exp\left\{ S_0[n] + \frac{\lambda}{2}\Big(\sum_x \vec{n}_x\Big)^2 - \sum_x \ln n_x^0 + 2\ln \sum_x n_x^0 \right\}, \qquad \text{(B.1)}$$

with $S_0[n]$ regarded as a functional of the 'spatial' components $\vec{n}_x$ of the n-field only, eliminating n_x^0 via $n_x^0 = \sqrt{1+\vec{n}^2}$. Note that, as in the case of gauge field theory, the form of the Faddeev-Popov term is identical in the delta-function and the smooth λ gauges.

We shall indicate the procedure for the case of the rotation Ward identity in (2.23) only – the derivation of the correction terms for the boost Ward identity is analogous but more tedious. We shall comment on it at the end of this Appendix. For the rotation Ward identity it is enough to consider a rotation in the n^1, n^2 plane and we may wlog consider the $\mathrm{SO}(1,2)$ model throughout. Performing a local rotation with angle α_x on the $\vec{n}_x$ field then gives

$$n_x \cdot n_{x+\hat\mu} \;\longrightarrow\; n_x^0 n_{x+\hat\mu}^0 - |\vec{n}_x||\vec{n}_{x+\hat\mu}| \cos(\theta_x - \theta_{x+\hat\mu} + \alpha_x - \alpha_{x+\hat\mu}) \qquad \text{(B.2a)}$$

$$\Big(\sum_x \vec{n}_x\Big)^2 \;\longrightarrow\; \Big(\sum_x |\vec{n}_x| \cos(\theta_x + \alpha_x)\Big)^2 + \Big(\sum_x |\vec{n}_x| \sin(\theta_x + \alpha_x)\Big)^2 \qquad \text{(B.2b)}$$

$$\vec{n}_x^1 \;:=\; |\vec{n}_x| \cos\theta_x, \;\; \vec{n}_x^2 := |\vec{n}_x| \sin\theta_x\,. \qquad \text{(B.2c)}$$

Introducing these transformations into (B.1) and expanding to second order in the α_x, one generates three sorts of terms, which we shall refer to henceforth as the "A", "B" and "C" type contributions to the rotation Ward identity. The A-terms arise from the variation of the $\beta n_x \cdot n_{x+\hat\mu}$ term in the action of (B.1), the B-terms from the λ gauge-fixing term,and the C terms are those quadratic terms arising as cross terms of the (linear) variation in the pure action and gauge-fixing term. Note that the Faddeev-Popov and nonlinear field measure terms play no role as they are invariant under rotations. The invariance of the functional integral under the change of variables (B.2) then implies

$A+B+C=0$ where, after some calculation, we find:

$$A = \frac{1}{2}\sum_{x\mu,y\nu}(\Delta_\mu\alpha_x)(\Delta_\nu\alpha_y)\langle J_{x\mu}J_{y\nu}\rangle - \frac{\beta}{2}\sum_{x\mu}(\Delta_\mu\alpha_x)^2\langle\vec{n}_x\cdot\vec{n}_{x+\hat{\mu}}\rangle\,, \tag{B.3a}$$

$$B = -\frac{\beta\lambda}{2}\Big\{\sum_{xy}\Big(\frac{\alpha_x^2+\alpha_y^2}{2}\langle n_x^1 n_y^1\rangle + \alpha_x\alpha_y\langle n_x^2 n_y^2\rangle\Big) - \beta\lambda\sum_{xyzw}\alpha_x\alpha_y\langle n_x^2 n_y^2 n_z^1 n_w^1\rangle\Big\}\,, \tag{B.3b}$$

$$C = -\beta\lambda\sum_{xyz\mu}(\Delta_\mu\alpha_x)\alpha_y\langle J_{x\mu}\, n_y^2 n_z^1\rangle\,, \tag{B.3c}$$

where as in Section 2

$$J_{x\mu} := J_{x\mu}^{12} = \beta(n_x^1 n_{x+\hat{\mu}}^2 - n_x^2 n_{x+\hat{\mu}}^1)\,.$$

At this point it is convenient to go over to momentum space by introducing discrete Fourier transforms appropriate for the lattice in question: thus, $\alpha_x = \frac{1}{V}\sum_p e^{ip\cdot x}\alpha(p)$ etc. One then finds for the A type contributions

$$A = -\frac{1}{2V}\sum_p\sum_\mu(2-2\cos p_\mu)[2\beta E^1 - \mathcal{J}_L(p)]\,\alpha(p)\alpha(-p)\,, \tag{B.4}$$

with $E^a = \langle n_x^a n_{x+\hat{\mu}}^a\rangle$, $a=1,2$, equal constants by translation and rotation invariance. Similarly, the B type terms may be rewritten

$$B = \frac{1}{V}\sum_p\alpha(p)\alpha(-p)\Big\{-\frac{\beta\lambda}{2}[D(p)+D(0)] + \frac{\beta^2\lambda^2}{2}\Gamma(p)\Big\}\,. \tag{B.5}$$

where we have defined two- and four-point functions D and Γ resp. as

$$\begin{aligned}\langle n_x^a n_y^b\rangle &:= \delta^{ab}D_{xy}, \quad D_{xy} = \frac{1}{V}\sum_p e^{ip\cdot(x-y)}D(p)\,,\\ \langle n_x^2 n_y^2\,\tilde{n}^1(0)\tilde{n}^1(0)\rangle &:= \frac{1}{V}\sum_p e^{ip\cdot(x-y)}\Gamma(p)\,,\end{aligned}$$

where the tilde notation indicates Fourier transform on the $\vec{n}$ fields $\tilde{n}^a(p) := \sum_x e^{ip\cdot x}n_x^a$. Finally, introducing the three-point function Ξ as follows (Δ_μ^* denotes a left lattice derivative)

$$\Xi_{xy} := \Big\langle(\Delta_\mu^* J_{x\mu}^0)\, n_y^2\tilde{n}^1(0)\Big\rangle = \frac{1}{V}\sum_p e^{ip\cdot(x-y)}\Xi(p)\,,$$

we find that the C type term amounts to

$$C = \beta\lambda\frac{1}{V}\sum_p\alpha(p)\alpha(-p)\Xi(p)\,. \tag{B.6}$$

To summarize, the *exact* rotation Ward identity in a λ-gauge takes the form

$$\sum_{\mu}(2-2\cos p_\mu)[2\beta E^1 - \mathcal{J}_L(p)] = \beta\lambda[2\Xi(p) - D(p) - D(0)] + \beta^2\lambda^2\Gamma(p)\,. \tag{B.7}$$

The terms arising from the gauge-fixing part of the action are isolated on the right-hand-side of (B.7): the left-hand-side corresponds precisely to the naive Ward-identity (2.30). To obtain the form of the Ward identity appropriate for the delta-function gauge used for the simulations, we must next examine the limit of the right-hand-side when $\lambda \to \infty$. In order to do this, we note that the action in (B.1) can be written $S = S_0 + \frac{\beta\lambda}{2}[\tilde{n}^1(0)^2 + \tilde{n}^2(0)^2]$. Furthermore, we have the distributional limit

$$e^{-\frac{\beta\lambda}{2}x^2} \;\longrightarrow\; \delta(x) + \frac{1}{2\beta\lambda}\delta''(x) + \frac{1}{8\beta^2\lambda^2}\delta''''(x) + \dots\,, \quad \text{as } \lambda\to\infty\,. \tag{B.8}$$

We shall momentarily use the notation $\langle\mathcal{O}\rangle_0 := \frac{1}{Z_0}\int d\vec{n}\,\delta(\sum_x \vec{n}_x)\,\mathcal{O}\,e^{-S_0}$ to denote the expectation of a Green's function $G(\vec{n})$ in the delta-gauge, whereas $\langle\,\cdot\,\rangle$ will denote the λ-gauge expectation, as previously. The B term (B.3b) in λ-gauge can be rewritten

$$B = -\frac{\beta\lambda}{2V}\sum_x \alpha_x^2\langle\tilde{n}^1(0)^2\rangle - \frac{1}{2}\beta\lambda\Big\langle\sum_{xy}\alpha_x\alpha_y\,n_x^2 n_y^2\Big(1-\beta\lambda\tilde{n}^1(0)^2\Big)\Big\rangle\,. \tag{B.9}$$

Using (B.8), it is easy to obtain for the infinite λ limit of the first term on the right-hand-side of (B.9)

$$-\frac{\beta\lambda}{2V}\sum_x \alpha_x^2\langle\tilde{n}^1(0)^2\rangle \;\longrightarrow\; -\frac{1}{2V}\sum_x\alpha_x^2 = -\frac{1}{2V^2}\sum_p \alpha(p)\alpha(-p)\,, \tag{B.10}$$

Again, using (B.8), one easily verifies, for any $\mathcal{O}$ independent of n^1, in the infinite λ limit:

$$\Big\langle\beta\lambda[1-\beta\lambda\tilde{n}^1(0)^2]\,\mathcal{O}(n^2)\Big\rangle \;\longrightarrow\; -\frac{3}{2}\Big\langle\mathcal{O}(n^2)\Big(\frac{\partial S_0}{\partial\tilde{n}^1(0)}\Big)^2 - \frac{\partial^2 S_0}{\partial\tilde{n}^1(0)^2}\Big)\Big\rangle_0\,. \tag{B.11}$$

The first derivative term in (B.11) can be written in terms of a new field ξ_x, as follows:

$$\begin{aligned}\frac{\partial S_0}{\partial\tilde{n}^1(0)} &= \frac{1}{V}\sum_x \xi_x\,, \qquad \text{with} \\ \xi_x &:= \frac{1}{n_x^0}\left(\beta\sum_\mu(n^0_{x+\hat{\mu}} + n^0_{x-\hat{\mu}}) + \frac{1}{n_x^0} - \frac{2}{\sum_x n_x^0}\right)n_x^1\,,\end{aligned} \tag{B.12}$$

while the second derivative term is easily seen to be suppressed by a factor of volume V relative to the first, and will be ignored henceforth. In momentum space, the B type terms are then found to yield

$$\frac{1}{V}\sum_p\Big\{-\frac{1}{2V} + \frac{3}{4V^3}\langle\tilde{n}^2(p)\tilde{n}^2(-p)\tilde{\xi}(0)^2\rangle\Big\}\,. \tag{B.13}$$

The large λ limit of the C type cross term can likewise be evaluated

$$\beta\lambda\sum_{xy}\alpha_x\alpha_y\langle\Delta^*_\mu J^0_{x\mu}n^2_y\tilde{n}^1(0)\rangle \quad\longrightarrow\quad -\frac{\beta}{V^2}\sum_p\sum_\mu(2-2\cos p_\mu)\alpha(p)\alpha(-p)D(p) -\frac{1}{V^2}\sum_p\alpha(p)\alpha(-p)\Xi'(p)\,,$$

with the modified three-point function

$$\Xi'(p) := \sum_x e^{ip\cdot(x-y)}\langle\Delta^*_\mu J^0_{x\mu}\, n^2_y\tilde{\xi}(0)\rangle\,. \tag{B.14}$$

Combining these results, we find that the rotational Ward identity, given by (B.7) in the smooth λ gauges, becomes in delta-function gauge

$$\sum_\mu(2-2\cos p_\mu)[2\beta E^1-\mathcal{J}_L(p)] = \tag{B.15}$$
$$-\frac{1}{V}-\frac{2\beta}{V}\sum_\mu(2-2\cos p_\mu)D(p)-\frac{2}{V}\Xi'(p)+\frac{3}{2V^3}\langle\tilde{n}^2(p)\tilde{n}^2(-p)\tilde{\xi}(0)^2\rangle\,.$$

For p fixed, the first two terms are manifestly of order $\frac{1}{V}$ for large V. The last two terms involve 3- and 4-point functions respectively, with zero-momentum insertions, for which the volume dependence is not a-priori clear. However, they may be computed easily from the numerical simulations of Section 4. We find that the best fits to the volume dependence of the last two terms suggest a behavior $\sim\frac{1}{V}\ln V$. These fits were performed using the results of measurements on 32^2, 64^2 and 128^2 lattices.

Finally, we show in Fig. 12 the right-hand-side of (B.15), divided by the trivial kinematic factor $\sum_\mu(2-2\cos p_\mu)$, for β =10 and L =32,64, and 128. In all cases, they represent a small numerical correction to the left-hand-side, as expected from the agreement found in Section 4.

Finite volume corrections to the boost Ward identity in (2.23) can be computed in a manner precisely analogous to the procedure leading to (B.15). Apart from a trivial $\frac{1}{V}$ term, there are in this case nine structures appearing on the right-hand-side of the Ward identity. As the formulas are somewhat lengthy we refrain from spelling them out here. However we have studied the finite volume dependence of these terms on 32^2, 64^2 and 128^2 lattices, and again find that the dominant asymptotic behavior is $\frac{1}{V}\ln V$, as for the rotation Ward identity.

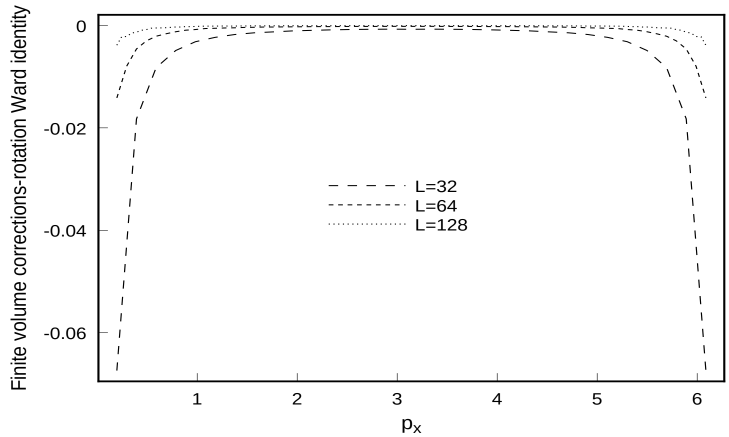

Figure 12: Volume dependence of correction terms to rotation Ward identity.

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